# Generalized Non-Homogeneous Morrey Spaces And Olsen Inequality

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## Abstract

In this paper, we shall discuss some properties of generalized non-homogeneous Morrey spaces. In addition, we will also prove the Olsen inequality in the non-homogeneous setting. Our proof utilizes the result of (García-Cuerva and Martell, 2001) on the boundedness of the fractional integral operator on Lebesgue spaces of non-homogeneous type.

**Keywords:** *Olsen inequality, fractional integral operator, non-homogeneous Lebesgue spaces, generalized non-homogeneous Morrey spaces.*

## 1. Introduction

We shall study here the fractional integral operator $I_\alpha^n$ (for $0<\alpha<n\le d$), on non-homogeneous spaces, which is defined by the formula

$$I_\alpha^n f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\mu(y) .$$

The formula reduced to the classical version of (Hardy and Littlewood, 1927; Hardy and Littlewood, 1932; and Sobolev, 1938) when $n=d$ and $\mu$ is the usual Lebesgue measure. By a non-homogeneous space we mean a metric space -- here we will consider only the Euclidean space $\mathbf{R}^d$ -- equipped with an $n$-dimensional measure (García-Cuerva and Martell, 2000). A positive Borel measure $\mu$ satisfies $n$-dimensional measure (for $0 < n \le d$) if there exists a constant $C>0$ such that

$$\mu(B(a,r)) \le Cr^n$$

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for every open ball $B(a,r)$ centered at $a \in \mathbf{R}^d$ with radius $r > 0$ (García-Cuerva and Gatto, 2004). This condition -- also known as the *growth condition* of order *n* (Sawano, 2005) -- replaces the *doubling condition*, which is the key property for a metric space to be a homogeneous space. Notice that a positive Borel measure $\mu$ satisfies the *doubling condition* if there exists a constant $C > 0$ such that for every ball $B(a,r)$ we have

$$\mu(B(a,2r)) \le C\mu(B(a,r))$$

(Coifman and Gusmán, 1970/1971). The ball $B(a,2r)$ is concentric to $B(a,r)$ with radius *2r*. We may consult (Krantz, 1999) for examples of the spaces of homogeneous type and (Verderra, 2002) for that of non-homogeneous type.

Now, let $L^p(\mu) = L^p(\mathbf{R}^d, \mu)$, $1 \le p < \infty$, denote the non-homogeneous Lebesgue spaces. It is well known from (García-Cuerva and Martell, 2001) that $I_\alpha^n$ is a bounded operator from $L^p(\mu)$ to $L^q(\mu)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Further, the following Olsen inequality

$$\left\| WI_\alpha f : L^p(\mu) \right\| \le C \left\| W : L^{n/\alpha}(\mu) \right\| \left\| f : L^p(\mu) \right\|,$$

for $W \in L^{n/\alpha}(\mu)$, can be viewed as a consequence of the $L^p(\mu) - L^q(\mu)$ boundedness of $I_\alpha^n$ (Sihwaningrum, *et.al.*, 2008b). The inequality was first introduced -- in homogeneous setting -- by (Olsen, 1995) to study the solution of the Schrödinger equation with a small perturbed potential *W* on Morrey spaces. Later on, (Kurata *et al*., 2002; Gunawan and Eridani, 2008) extended the Olsen's result to the homogeneous generalized Morrey spaces. In this paper, we will extend further the Olsen's result to the generalized Morrey spaces of non-homogeneous type.

## 2. Main Results

### 2.1 Generalized non-homogeneous Morrey spaces

For $1 \le p < \infty$ and $\phi : (0,\infty) \to (0,\infty)$, let us define the generalized non-homogeneous Morrey spaces $\mathrm{M}^{p,\phi}(\mu) = \mathrm{M}^{p,\phi}(\mathbf{R}^d, \mu)$ to be the set of all functions $f \in L^p_{loc}(\mu)$ for which $\left\| f : \mathrm{M}^{p,\phi}(\mu) \right\| < \infty$. Here,

$$\left\| f : M^{p,\varphi}(\mu) \right\| := \sup_{r>0} \frac{1}{\phi(r)} \left( \frac{1}{r^n} \int_{B(x,r)} \left| f(y) \right|^p d\mu \right)^{\frac{1}{p}}, \qquad 1 \le p < \infty .$$

Our definition is in line with the definition of Hardy-L ittlewood maximal operator $M^n$ given by the formula

$$M^n f(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(a,r))} \left| f(y) \right| d\mu(y) .$$

The reader may also refer (Gunawan, *et.al.,* 2007) and (Sawano, 2008) for other types of generalized non-homogeneous Morrey spaces – which are defined in accordance with the *k*-dilated Hardy-Littlewood maximal operator $M_k$ :

$$M_k f(x) := \sup_{Q \ni x} \frac{1}{\mu(kQ)} \int_Q f(y)\, d\mu(y)$$

Note that along with our definition, if $\phi(r) = r^{-n/p}$, we obtain $M^{p,\phi}(\mu) = L^p(\mu)$. Meanwhile, for $1 < p < q < \infty$, we have $M^{q,\phi}(\mu) \subseteq M^{p,\phi}(\mu) \subseteq M^{1,\phi}(\mu)$ (see (Sihwaningrum, *et. al.,* 2008a) for the proof). Furthermore, the generalized non-homogeneous Morrey spaces obey the following property.

**Fact 2.1** *If* $1 \le p < \infty$ *and* $\phi(r) \le C\psi(r)$ *(for* $r > 0$*), then* $M^{p,\psi}(\mu) \subseteq M^{p,\phi}(\mu)$ *and* $\left\| f : M^{p,\psi}(\mu) \right\| \le C \left\| f : M^{p,\phi}(\mu) \right\|$.

**Proof.** Notice first that $\frac{1}{\psi(r)} \le C \frac{1}{\phi(r)}$ for all $r > 0$. Then, for $f \in M^{p,\phi}(\mu)$, we get

$$\sup_{r>0} \frac{1}{\psi(r)} \left( \frac{1}{r^n} \int_{B(x,r)} \left| f(y) \right|^p d\mu \right)^{\frac{1}{p}} \le C \sup_{r>0} \frac{1}{\phi(r)} \left( \frac{1}{r^n} \int_{B(x,r)} \left| f(y) \right|^p d\mu \right)^{\frac{1}{p}} < \infty .$$

The above inequality implies that $\left\| f : M^{p,\psi}(\mu) \right\| \le C \left\| f : M^{p,\phi}(\mu) \right\|$, and so does $M^{p,\psi}(\mu) \subseteq M^{p,\phi}(\mu)$. □

As a consequence, $M^{p,\psi}(\mu) = M^{p,\phi}(\mu)$ and $\left\| f : M^{p,\psi}(\mu) \right\| \sim \left\| f : M^{p,\phi}(\mu) \right\|$ for $\phi \sim \psi$.

## 2.2 Olsen Inequality

As an extention of the Olsen's result, we will present here an Olsen inequality on generalized non-homogeneous Morrey spaces. The inequality simply says that a multiplication of operators $W$ and $I_\alpha^n$ is bounded on $M^{p,\phi}(\mu)$. To proof the inequality, we use the the boundedness of $I_\alpha^n$ from $L^p(\mu)$ to $L^q(\mu)$.

**Theorem 2.2.** *Suppose that $\phi$ satisfies the doubling condition for function, that is there exists a constant C such that* $\frac{1}{2} \le \frac{t}{s} \le 2 \Rightarrow \frac{1}{C} \le \frac{\phi(t)}{\phi(s)} \le C$. *Suppose further that $\phi$ satisfies* $\int_r^\infty t^{\alpha-1}\phi(t)dt \le Cr^\alpha \phi(r)$. *Then, the inequality*

$$\left\| WI_\alpha f : M^{p,\phi}(\mu) \right\| \le C \left\| W : L^{n/\alpha}(\mu) \right\| \left\| f : M^{p,\phi}(\mu) \right\|$$

*holds provided that* $W \in L^{n/\alpha}$.

**Proof**. Let $B := B(a,r)$ dan $\hat{B} := B(a,2r)$ where $a \in \mathbf{R}^d$. Then, we decompose the function $f \in M^{p,\phi}(\mu)$ as $f = f_1 + f_2 = f_{\chi_{\hat{B}}} + f_{\chi_{\hat{B}^C}}$. Recall that $I_\alpha^n$ is a bounded operator from $L^p(\mu)$ to $L^q(\mu)$, so that we have

$$\left( \frac{1}{r^n} \int_B \left| I_\alpha^n f_1(x) \right|^q d\mu(x) \right)^{1/q} \le C\psi(r) \left\| f : M^{p,\phi}(\mu) \right\|.$$

Now, by using the Hölder inequality and $L^p(\mu)$ - $L^q(\mu)$ boundedness of $I_\alpha^n$, we obtain

$$\left( \int_B \left| WI_\alpha^n f_1(y) \right|^p d\mu(y) \right)^{1/p} \le Cr^{n/p}\phi(r) \left\| W : L^{n/\alpha}(\mu) \right\| \left\| f : M^{p,\phi}(\mu) \right\| . \tag{2.1}$$

As the measure $\mu$ satisfies the growth condition, then for every $x \in B$ we could find the estimate

$$\left| I_\alpha^n f_2(x) \right| \le \int_{|x-y| \ge r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y) \le C \left\| f : M^{p,\phi}(\mu) \right\| \sum_{k=0}^\infty \left(2^k r\right)^\alpha \phi\left(2^k r\right).$$

We see that the right hand side of the inequality contains the summation from $k=0$ to $k=\infty$. So, we utilize the doubling condition of $\phi(t)$ and $t^{\alpha}$ to get

$$\left(2^{k}r\right)^{\alpha}\phi\left(2^{k}r\right)\leq C\int_{2^{k-1}}^{2^{k}}t^{\alpha-1}\phi(t)dt$$

for $k=0,1,2,\ldots$. As a result

$$\left|I_{\alpha}^{n}f_{2}(x)\right|\leq C\left\|f:M^{p,\phi}(\mu)\right\|\sum_{k=0}^{\infty}\int_{2^{k-1}}^{2^{k}}t^{\alpha-1}\phi(t)dt\leq Cr^{\alpha}\phi(r)\left\|f:M^{p,\phi}(\mu)\right\|.$$

Furthermore, the Hölder inequality allows us to obtain

$$\left(\frac{1}{r^{n}}\int_{B}\left|WI_{\alpha}^{n}f_{1}(y)\right|^{p}d\mu(y)\right)^{1/p}\leq Cr^{n/p}\phi(r)\left\|W:L^{n/\alpha}(\mu)\right\|\left\|f:M^{p,\phi}(\mu)\right\|. \qquad (2.2)$$

Now, we apply the Minkowski inequality to the estimate (2.1) and (2.2) to get

$$\begin{aligned}&\frac{1}{\phi(r)}\left(\frac{1}{r^{n}}\int_{B}\left|WI_{\alpha}^{n}f(y)\right|^{p}d\mu(y)\right)^{1/p}\\&\leq\frac{1}{\phi(r)}\left(\frac{1}{r^{n}}\int_{B}\left|WI_{\alpha}^{n}f_{1}(y)\right|^{p}d\mu(y)\right)^{1/p}+\frac{1}{\phi(r)}\left(\frac{1}{r^{n}}\int_{B}\left|WI_{\alpha}^{n}f_{2}(y)\right|^{p}d\mu(y)\right)^{1/p}\\&\leq C\left\|W:L^{n/\alpha}(\mu)\right\|\left\|f:M^{p,\phi}(\mu)\right\|\end{aligned}$$

By taking the supremum over all $r>0$, we complete our proof. □

### 3. Concluding Remarks

We may also proof Theorem 2.2 by using the $M^{p,\phi}(\mu)$ - $M^{q,\psi}(\mu)$ boundedness of $I_{\alpha}^{n}$, where $1<p<q<\infty$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and $r^{\alpha}\phi(r)\leq C\psi(r)$. The alternative proof is simpler than that of presented here (see (Sihwaningrum, *et. al.,* 2008b) for detail) .

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