The Properties of Fuzzy Green Relations on Bilinear Form Semigroups

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Abstract—The Green relations on semigroups have been introduced by Howie [3]. They are right Green relation \( R \), left Green relation \( L \) and (two sided) Green relation \( I \). The right Green relation \( R \) is defined as \( \{(x,y) \in S \times S|\langle x \rangle_R = \langle y \rangle_R\} \), with \( \langle x \rangle_R \) denotes the right ideal generated by an element \( x \) (or called the principle right ideal generated by \( x \)). The definition of the left Green relation \( L \) and the Green relation \( I \) are similar to the definition of the right Green relation. In this paper we will construct the definition of the fuzzy right Green relation (denoted by \( R^f \)), the fuzzy left Green relation (denoted by \( L^f \)) and the fuzzy Green relation (denoted by \( I^f \)) on a semigroup. First we define a fuzzy ideal (right/left) generated by a fuzzy subset (a fuzzy principle ideal) on a semigroup and their examples. Based on the fuzzy principle ideal definition, we define a fuzzy (right/left) Green relation on a semigroup. The fuzzy subset \( \mu \) and \( \rho \) are fuzzy (right/left) Green related if and only if the fuzzy (right/left) ideal generated by \( \mu \) is equal to the fuzzy (right/left) ideal generated by \( \rho \).

Keywords—Green relation, fuzzy ideal, fuzzy principal ideal, fuzzy Green relation

I. Introduction

A non empty subset \( I \) of a semigroup \( S \) is called a right (left) ideal if \( IS \subseteq I \) (SI \( \subseteq I \)) and an ideal (two sided) if \( I \) is both a right ideal and a left ideal. The right (left) ideal generated by \( x \in S \) is denoted by \( \langle x \rangle_R \) (\( \langle x \rangle_L \)) and an ideal generated by \( x \in S \) is denoted by \( \langle x \rangle \). The Green relation on a semigroup has been introduced by Howie [3]. They are right Green relation \( R \), the left Green relation \( L \) and the Green relation \( I \). The Green relation \( R, L, I \) are equivalence relations, defined as follow:

\[
R = \{(x,y) \in S \times S|\langle x \rangle_R = \langle y \rangle_R\}
\]

\[
L = \{(x,y) \in S \times S|\langle x \rangle_L = \langle y \rangle_L\}
\]

\[
I = \{(x,y) \in S \times S|\langle x \rangle = \langle y \rangle\}
\]

Some papers related to the fuzzy ideal of semigroups, the fuzzy ideal of semigroups generated by a fuzzy singleton and their properties have been introduced by Karyati [5]. In this paper we will discuss how to define the fuzzy Green relations on a semigroup based on the fuzzy (right/left) ideal generated by a fuzzy subset of this semigroup.

II. Fuzzy Green Relations on Semigroup

Refer to Asaad [2], Kandasamy [4], Mordeson and Malik [7], a fuzzy subsemigroup \( \mu \) of a semigroup \( S \) is defined as a mapping from \( S \) into the interval [0,1], i.e. \( \mu:S \rightarrow [0,1] \) which fulfills the condition \( \mu(xy) \geq \min\{\mu(x),\mu(y)\} \) for all \( x, y \in S \). A fuzzy subset \( \mu \) is called a fuzzy right (fuzzy left) ideal of \( S \) if \( \mu(x) \geq \mu \) for every \( x, y \in S \) then \( \mu(xy) \geq \mu(x)(\mu(xy) \geq \mu(x)) \) and \( \mu \) is called fuzzy ideal of \( S \) if \( \mu(x) \) is both a fuzzy right ideal and a fuzzy left ideal, i.e. \( \mu(xy) \geq \max\{\mu(x),\mu(y)\} \) for all \( x, y \in S \). Fuzzy subsets \( \lambda \) and \( \mu \) are called \( \lambda \subseteq \mu \) if and only if \( \lambda(x) \leq \mu(x) \) for every \( x, y \in S \). A fuzzy relation \( \theta \) of \( S \) is defined as a mapping from \( S \times S \) into the closed interval [0,1].

Definition 2.1. ([1], [6], [9]) Let \( S \) be a semigroup and \( \mu \) be a fuzzy relation on \( S \). Then

1. A fuzzy relation \( \mu \) on \( S \) is said to be reflexive if \( \mu(x,x) = 1 \) for all \( x \in S \)
2. A fuzzy relation \( \mu \) on \( S \) is said to be symmetric if \( \mu(x,y) = \mu(y,x) \) for all \( x, y \in S \)
3. If \( \mu_1 = \mu_2 \) are two relations on \( S \), then their max-product composition denoted by \( \mu_1 \circ \mu_2 \) is defined as \( \mu_1 \circ \mu_2(x,y) = \max_{z \in S}\{\mu_1(x,z), \mu_2(z,y)\} \)
4. If \( \mu_1 = \mu_2 = \mu \) and \( \mu \circ \mu \leq \mu \), then the fuzzy relation \( \mu \) is called transitive.

Refer to Aktas [1], Kuroki [6], and Murali [9], we give some kinds of relations defined as follow: