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A Numerical Method For Infiltration Problems

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Abstract

In this paper, an infiltration problem is considered. The problem involves time-independent flow from periodic channels in a homogeneous soil. A governing equation of the problem is the Richards equation. This equation can be studied more conveniently by transforming the equation to a modified Helmholtz equation using the Kirchhoff transformation, dimensionless variables and a Laplaces transform. This modified Helmholtz equation may be solved numerically using a dual reciprocity boundary element method (DRBEM).

1 Introduction

In the last few decades, problems involving water infiltration in porous medium have been studied by a number of researchers. Waechter and Mandal studied steady infiltration from a semicircular cylindrical trench and hemispherical pond [11]. Steady infiltration from buried and surface cavities have been considered by Pullan and Collins [8]. Steady infiltration problems from irrigation channels have been investigated by Batu [3], Azis et al [2], Clements et al [5], and Lobo et al [7]. For problems involving infiltration from irrigation channels, Batu solved infiltration problems analytically. Azis et al, Clements et al, and Lobo et al solved infiltration problems using standard Boundary Element Methods (BEM). In this paper, we will demonstrate a numerical method, which is known as Dual Reciprocity Boundary Element Method (DRBEM) to solve a steady infiltration problem. Numerical results

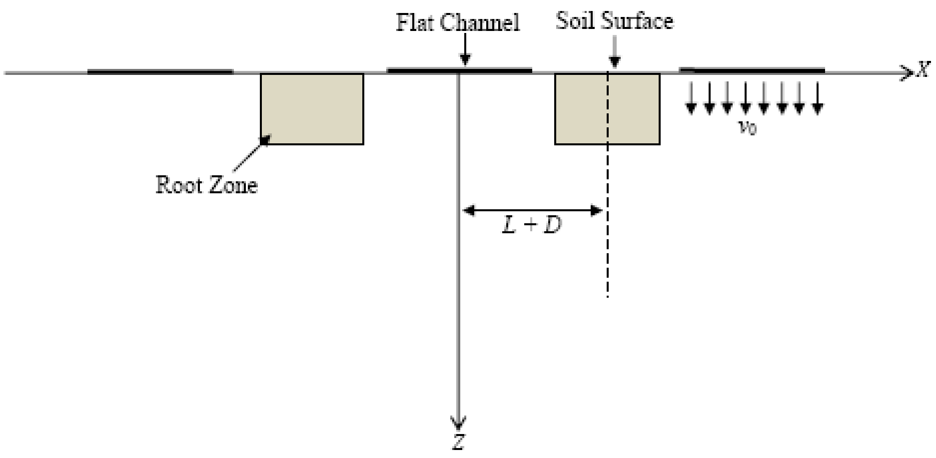

Figure 1: Geometry of periodic flat channels with roots zone

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2 Problem Formulation

Using a Cartesian coordinate system $OXYZ$ with OZ vertically positive downwards, we consider a type of homogeneous soil in the region $Z \geq 0$. Periodic identical flat channels, channels without depth, are created on the surface of the soil. The width of the channels is $2L$. Crops, with roots of depth Z_m and width $2X_m$, are planted between and equidistant from two adjacent channels. The distance between two consecutive rows of crops is $2(L+D)$.

It is assumed that the geometries of the channels and the root distribution in the root zone do not vary in the OY direction and are symmetrical about the planes $X = \pm k(L+D)$, for $k = 0, 1, 2, \cdots$. Hence, the problem may be simplified to two dimensions, which is illustrated in Figure 1. Water is supplied from the channels in uniform fluxes, v_0. However, on the soil surface outside the channels, the flux is zero. Given this situation, we wish to determine suction potential in the homogeneous soil type stated above.

Because of the symmetry of the problem, it is sufficient to consider the semi infinite region defined by $0 \leq X \leq L+D$ and $Z \geq 0$. This region is denoted by R with boundary C. The boundary along the surface of the channel is denoted by C_1 and the surface of soil outside the channel by C_2. The boundary along $X = L+D$ is denoted by C_3, and C_4 is for $X = 0$. The fluxes over C_1 is v_0, while over C_2 is 0. There are no fluxes across C_3 and C_4 as the problem symmetrical about them. The derivatives $\partial\Theta/\partial X \to 0$ and $\partial\Theta/\partial Z \to 0$ as $X^2 + Z^2 \to \infty$ [Batu].

3 Basic Equations

The governing equation of steady infiltration with root water uptake is given by

$$\frac{\partial}{\partial X}\left(K\frac{\partial\psi}{\partial X}\right) + \frac{\partial}{\partial Z}\left(K\frac{\partial\psi}{\partial Z}\right) - \frac{\partial K}{\partial Z} = S(X, Z, \psi), \tag{1}$$

where K is the hydraulic conductivity, ψ is the suction potential, and S is the root water uptake function.

The relation between K and ψ is defined as

$$K = K_s e^{\alpha\psi}, \quad \alpha > 0, \tag{2}$$

where α is an empirical parameter and K_s is the saturated hydraulic conductivity. The Matric flux potential (MFP), Θ, is obtained using the Kirchhoff transformation

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$$\Theta = \int_{-\infty}^{\psi} K ds. \tag{3}$$

Using equations (2) and (3), the following equation is obtained

$$\psi = \frac{1}{\alpha} \ln \left(\frac{\alpha \Theta}{K_s} \right), \tag{4}$$

and Equation (1) is transformed to

$$\frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Z^2} - \alpha \frac{\partial \Theta}{\partial Z} = S \left(X, Z, \frac{1}{\alpha} \ln \left[\frac{\alpha \Theta}{K_s} \right] \right). \tag{5}$$

The flux normal to the surface with outward pointing normal $\mathbf{n} = (n_1, n_2)$ is given by

$$F = -\frac{\partial \Theta}{\partial X} n_1 + \left(\alpha \Theta - \frac{\partial \Theta}{\partial Z} \right) n_2. \tag{6}$$

The root water uptake function is as that in [?], which takes the form

$$S(X, Z, \psi) = \gamma(\psi) \frac{L_t \beta(X, Z) T_{pot}}{\int_0^{Z_m} \int_{L+D-X_m}^{L+D} \beta(X, Z) dX dZ}, \tag{7}$$

where γ is the root-water stress response function, L_t is the width of soil surface associated with the transpiration process, T_{pot} is the potential transpiration, and $\beta(X, Z)$ is the two-dimensional spatial root distribution, which takes the form

$$\begin{aligned} \beta(X, Z) \;=\; & \left(1 - \frac{L + D - X}{X_m} \right) \left(1 - \frac{Z}{Z_m} \right) \\ & \times e^{-(p_Z / Z_m |Z^* - Z| + p_X / X_m |X^* - (L+D-X)|)}, \\ & L + D - X_m \le X \le L + D, \quad 0 \le Z \le Z_m, \end{aligned} \tag{8}$$

where p_Z, p_X, X^*, and Z^* are empirical parameters.

Using the dimensionless variables

$$x = \frac{\alpha}{2} X, \quad z = \frac{\alpha}{2} Z, \quad \Phi = \frac{\Theta}{v_0 L}, \quad f = \frac{2\pi}{v_0 \alpha L} F, \tag{9}$$

and the transformation

$$\Phi = e^z \phi, \tag{10}$$

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equation (5) may be transformed to

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=\phi+\gamma^*(\phi)s^*(x,z)e^{-z}, \tag{11}$$

where

$$s^*(x,z)=\frac{2\pi}{\alpha L}\frac{l_t\beta^*(x,z)}{\int_0^{z_m}\int_{b-x_m}^{b}\beta^*(x,z)dxdz}\frac{T_{pot}}{v_0}, \tag{12}$$

and

$$\gamma^*(\phi)=\gamma\left(\frac{1}{\alpha}\ln\left(\frac{\alpha v_0 L\phi e^z}{\pi K_s}\right)\right). \tag{13}$$

Here

$$\begin{aligned}\beta^*(x,z) &= \left[1-\frac{b-x}{x_m}\right]\left[1-\frac{z}{z_m}\right]\\ &\quad\times e^{-(p_z/z_m|2z^*/\alpha-2z/\alpha|+p_x/x_m|2x^*/\alpha-2/\alpha\,(b-x)|)},\\ &\quad b-x_m\le x\le b,\quad 0\le z\le z_m.\end{aligned} \tag{14}$$

Parameters appear in Equations (12) and (14) are as follows.

$$\begin{aligned} l_t &= \frac{\alpha}{2}L_t,\quad x_m=\frac{\alpha}{2}X_m,\quad z_m=\frac{\alpha}{2}Z_m,\quad x^*=\frac{\alpha}{2}X^*,\quad z^*=\frac{\alpha}{2}Z^*\\ p_x &= \frac{\alpha}{2}p_X,\quad p_z=\frac{\alpha}{2}p_Z,\quad a=\frac{\alpha}{2}L\quad b=\frac{\alpha}{2}(L+D),\end{aligned} \tag{15}$$

Equations (6), (9), and (10) yield

$$f=-e^z\left[\frac{\partial\phi}{\partial x}n_1-\left(\phi-\frac{\partial\phi}{\partial z}\right)n_2\right]. \tag{16}$$

From equation (16), the normal derivative of ϕ may take the form

$$\frac{\partial\phi}{\partial n}=\phi n_2-e^{-z}f. \tag{17}$$

Boundary conditions in term of ϕ are summarized as follows.

$$\frac{\partial\phi}{\partial n} = \frac{2}{\alpha L}-\phi,\quad 0\le x\le\frac{\alpha L}{2}\text{ and } z=0, \tag{18}$$

$$\frac{\partial\phi}{\partial n} = -\phi,\quad \frac{\alpha L}{2}\le x=\le\frac{\alpha(L+D)}{2}\text{ and } z=0, \tag{19}$$

$$\frac{\partial\phi}{\partial n} = 0,\quad x=0\text{ and } z\ge 0, \tag{20}$$

$$\frac{\partial\phi}{\partial n} = 0,\quad x=b\text{ and } z\ge 0, \tag{21}$$

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and

$$\frac{\partial \phi}{\partial n} = -\phi, \text{ as } z \to \infty. \quad (22)$$

An integral equation to solve equation (11), as discussed by Ang [?], is

$$\begin{aligned}\lambda(\xi,\eta)\phi(\xi,\eta) &= \int\int_R \varphi(x,z;\xi,\eta)[\phi(x,z) + \gamma^*(\phi)s^*(x,z)e^{-z}]dxdz \\ &+ \int_C \left[\phi(x,z)\frac{\partial}{\partial n}(\varphi(x,z;\xi,\eta)) \right. \\ &- \left. \varphi(x,z;\xi\eta)\frac{\partial}{\partial n}(\phi(x,z))\right] ds(x,z), \quad (23)\end{aligned}$$

where

$$\lambda(\xi,\eta) = \begin{cases} \frac{1}{2}, & (\xi,\eta) \text{ lies on a smooth part of } C \\ 1, & (\xi,\eta) \in R \end{cases}, \quad (24)$$

and

$$\varphi(x,z;\xi,\eta) = \frac{1}{4\pi}\ln[(x-\xi)^2 + (z-\eta)^2] \quad (25)$$

is the fundamental solution of the Laplace's equation.

Integral equation (23) may be solved numerically using the dual reciprocity boundary element procedure with predictor corrector scheme as discussed in [9].

4 Results and Discussion

The method described in the preceding section is tested on problems involving infiltration from periodic flat channels into a homogeneous soil. The soil parameter, α, chosen in this study is that chosen by Lobo []. The value of α is 0.002 cm^{-1}. We first solve an infiltration problem without root-water uptake. Here, L and D are the same as those in [], 100 cm and 400 cm respectively.

To apply the DRBEM, domain must be bounded by a simple closed curve. Hence, an imposed boundary is needed. The domain is set to be between $z = 0$ and $z = 4$, sufficient depth for boundary conditions to be applied without significant impact to values of Φ in the domain. The boundary is then discretized into a number of elements, and a number of interior points is chosen. Let N and M be the numbers of elements and interior points respectively. We apply the DRBEM using two different sets of the pair of N and M. Let A and B be the two sets. In set A, the values of N and M are 400 and 625 respectively. The values of N and M in set B are 200 and 745 respectively. Numerical results obtained using the two sets are compared to those obtained using a standard BEM with 1000 elements and the analytic method. The results are presented in Table 1.

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From Table 1, it can be seen that numerical results obtained using Set A are more accurate than those obtained using Set B. This means that a higher number of elements gives better accuracy than a lower number of elements. It can also be seen that a standard BEM gives more accurate solutions than the DRBEM. In the construction of the BEM, solution of the problem is in a boundary integral. The boundary integral is then approximated by integral over elements. In the other hand, when constructing the DRBEM, solution of the problem is expressed in a boundary and a domain integral. The boundary integral is then treated as in the BEM. The domain integral is approximated by boundary integrals, and then the boundary integrals are treated as the BEM as before. Nevertheless, the numerical results obtained using the DRBEM are tolerable.

Table 1: Numerical solutions and analytic solutions at a number of points

Points	DRBEM (A)	DRBEM (B)	BEM (1000 Elements)	Analytic
(0.5,0.3)	0.9063	0.9082	0.9071	0.9070
(0.5,0.4)	0.9443	0.9464	0.9441	0.9440
(0.5,0.7)	0.9902	0.9934	0.9884	0.9884
(0.5,0.8)	0.9953	0.9988	0.9932	0.9932
(0.5,1.0)	1.0002	1.0046	0.9976	0.9976
(0.5,1.1)	1.0012	1.0063	0.9986	0.9986

Eventhough the standard BEM gives more accurate results, the BEM can only be applied for a limited number of problems. To apply the BEM, the fundamental solution of the governing equation is needed. However, it is not an easy task to obtain the fundamental solution. For instance, we may not be able to obtain the fundamental solution of Equation (5). Hence, to solve the equation, we use the DRBEM, as the method requires only the fundamental solution of Laplace equation.

Now, the DRBEM described in the preceding section is tested on problems involving infiltration from the periodic identical flat channels described above with root water uptake process. The potential transpiration rate, T_{pot}, is 4 cm/day. This transpiration rate was used by Li et al [6], and Šimunek and Hopmans [?] in their studies. The value of K_s in this study is 5.3 cm/day, which is the saturated hydraulic conductivity of Peat [1, 4]. The value of α of this soil is 0.00104 cm^{-1}, which is the closest value to the value of α used in this study.

The root-water stress response function γ used here is identical to that reported by Utset et al [10]. The value of h_4 in this report is -1120, as the value of ψ in the root zone less than -820. The value of h_3 for T_{pot} = 0.4 cm/d is interpolated from $h_{3,a}$ and $h_{3,b}$, and we have $h_3 = -470$.

The value of both X_m and Z_m in this study is 250 cm. The parameter values for the root distribution, $\beta(X, Z)$, are $X^* = 125$ cm, $P_X = 2.00$, $Z^* = 50$ cm and $P_Z = 5.00$. Using the DRBEM with the predictor corrector with $N = 200$ and $M = 742$, distribution of root uptake in the root zone is shown in Figure 2.

From Figure 2, it can be seen that the highest root uptake occurred in the region near the top soil and the plants. This result is expected, as the densest root distribution is in that region.

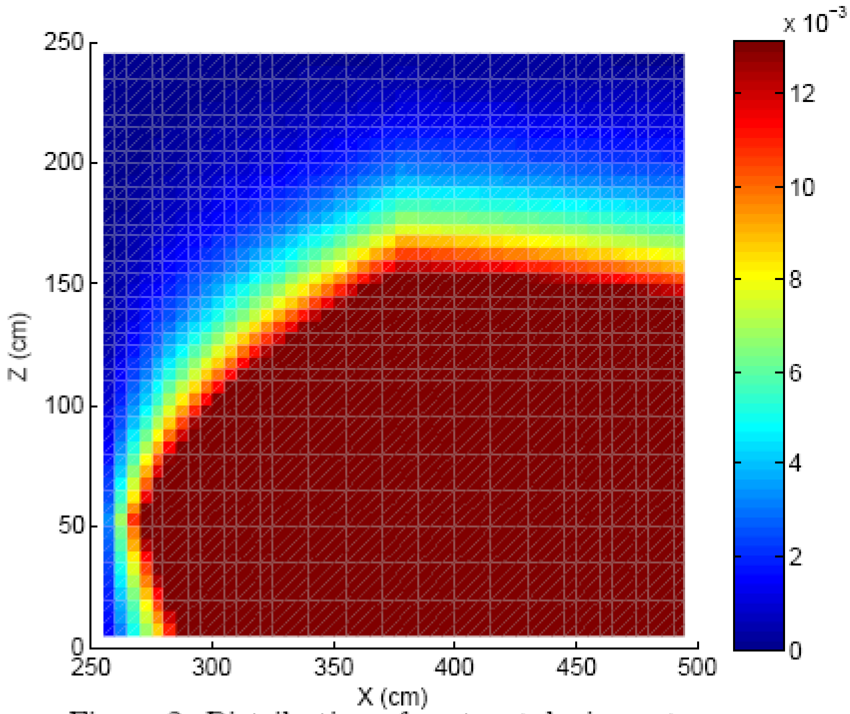

Figure 2: Distribution of root uptake in root zone.

5 Concluding Remarks

A Dual Reciprocity Boundary Element Method (DRBEM) has been formulated and successfully implemented to solve infiltration problems from periodic flat channels. For infiltration from flat channels without root water uptake, the number of line segments or elements affect the accuracy of the method more than the number of collocation interior points. However, a higher number of elements results in a longer computational time. Hence, one need to balance accuracy and efficiency.

For infiltration from flat channels with root water uptake, water uptake in root zone is investigated. The results indicate that the highest uptake occurs at the region near the surface of the soil and the plants.

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