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On the Pseudo-spectral Methods for Solving MKdV Equation

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Abstract

This paper will discuss about solving the modified-KdV equation numerically. Modified-KdV equation or mKdV equation is a KdV-type equation which has an exact dispersion relation. A well-known pseudo-spectral methods is chosen because the implementation can easily deal with the exact dispersion relation. The exact dispersion described by a non-polynomial pseudo-differential operator that can easily be dealt with in spectral space. An appropriate scheme is designed to predict wave evolutions using influxing technique. By using the Duhamel's principle, a homogeneous boundary value problem transformed into inhomogeneous problem.. We also compare the measurements from experiments and mKdV simulation result. The result show that the wave profile of mKdV equation has the same pattern with the measurement data.

Keywords :mKdV equation, dispersive waves, pseudo-spectral methods.

1. Introduction

Korteweg de Vries equation (KdV equation) is derived in 1985 as an equation which described travelling waves phenomenon [1]. This equation described a mass of water that travelling along a channel apparently without change of form or diminution of speed. In 1993, the mKdV equation, a new KdV-type equation is derived [1]. This equation is exact up to and including quadratic nonlinear terms and has exact dispersive properties. mKdV equation has been applied to study surface waveevolutions in [2].mKdV equation is derived from Boussinesq-type equation by using unidirectionalisation process.

We consider a fluid with condition irrotational flow, inviscidand incompressible that propagate in the $\mathbf{x} = (x, y)$ directionover a finite depth h_0. We denote thewave elevation by $\eta(x, t)$and the fluid potential by $\Phi(x, z, t)$with$\varphi(x, z = \eta, t)$the fluid potential at the surface.In [4], the dynamic equations canbe derived from variations of the action principle, $\delta A(\eta,\varphi) = 0$,

$$A(\eta,\phi) = \int\left(\int \phi\eta_t dx - H(\eta,\phi)\right)dt \tag{1.1}$$

The variation of $A(\eta,\varphi)$in equation (1.1)with respect to η and ϕ is given by

$$\partial_t \eta = \delta_\phi H(\eta,\phi)$$
$$\partial_t \phi = -\delta_\eta H(\eta,\phi)$$

According to [5], the Hamiltonian $H(\eta,\phi)$ is the total energy which expressed in η and ϕ.The total energy of the systems is the sum of the potential energy and kinetic energy. The potential energy is calculated with respect to the undisturbed water level,

$$H(\eta,\varphi) = \frac{1}{2}\int g\eta^2 dx + K(\eta,\varphi)$$

where

$$K(\eta,\varphi)=\frac{1}{2}\int_D |\nabla\phi|^2\,dxdz$$

We can write the dynamic equation as a Hamiltonian systems (see [2]) :

$$\partial_t\begin{pmatrix}\varphi\\ \eta\end{pmatrix}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta_\varphi H\\ \delta_\eta H\end{pmatrix}.$$

Since the kinetic energy cannot expressed explisitly, in [2, 5], the kinetic energy is approximated. By using unidirectionalisation process as in [2] we get

$$\eta_t+i\Omega(-i\partial_x)\eta+\frac{3}{4}\partial_x\eta^2=0 \quad (1.2)$$

where

$$\hat{\Omega}(k)=\sqrt{gh}k\sqrt{\frac{\tanh kh}{kh}} \quad (1.3)$$

In equations(1.2), Ω is pseudodifferentialoperator. According to [5], the pseudodifferential operator are a generalization of differential operators which acting upon a function as the inverse Fourier transform of a polynomial in the Fourier variable times the Fourier transform of the function. The pseudodifferential operator can be approximated by expanding $\hat{\Omega}(k)$ as in equations (1.3) using Taylor series in Fourier space and apply the invers Fourier transform into it.

Since the analytical solution of mKdV equation is difficult to obtained (see [2]), so the solution of mKdV equation will be approximated numerically. The main concern of this paper is to develop a spectral approximation of the mKdV equation. By using pseudospectral methods, the solution of mKdV equation can be done easily since the pseudodifferential operator can be dealt with in spectral space (see [6]). This spectral approximation is implemented in Matlab, and will be compared with experimental data.

2. Mathematical Interpretation of Given Signals

In this paper, we will compare the solution of mKdV equations with measurement from the experiment, so that we have to embedded a source function to mKdV equations(see [5]). By using influxing technique as in [5] the source function is choosen which has a dispersive property as the following

$$S(x,t)=\gamma(x).s(t),$$

where

$$\gamma(x)=\int\hat{\gamma}(x)e^{ikx}dk$$

and $s(t)$ is a given signal. The dispersive properties is given by

$$\hat{\gamma}(x)=V(k)=\frac{d\Omega}{dk}.$$

The condition of the source function are the following equations

$$\eta_t+i\Omega(-i\partial_x)\eta+\frac{3}{4}\partial_x\eta^2=0$$
$$\eta(0,t)=s_{flap}(t)$$
$$\eta(x,0)=0$$
$$\eta(beach,t)=0$$

At $x=0$ position, the wave generator gives a signal (say $s_{flap}(t)$). At the time $t=0$is the still water condition. At the end of the tank (wave tank in hydrodynamics laboratory) there is a beach or absorber. By using Duhamel's principle, mKdV equation became

$$\partial_t \eta + i\Omega\left(-i\partial_x\right)\eta + \frac{3}{4}\partial_x \eta^2 = s(x,t) \qquad (2.1)$$

where $S\left(x,t\right)$isthe source function. Equations (2.1) ismKdV equation with given signal. In this paper, the source functionbichromatics signal.

3. Spectral Implementation

Consider $\eta_N\left(x,t\right)$is the of mKdV equation as in equations (2.1)

$$\eta_N\left(x,t\right) = \sum_{k=-N/2}^{N/2} \hat{\eta}_k e^{ikx}. \qquad (3.1)$$

In equations(3.1), $\hat{\eta}_k$ is afunction of t. Now, to improve the calculation the mKdV equation as in (2.1) can be written as

$$\frac{\partial}{\partial t}\eta + i\Omega\left(-i\partial_x\right)\eta + \frac{3}{2}\eta\frac{\partial}{\partial x}\eta = s(t). \qquad (3.2)$$

Inserting u_N into the mKdV equations as inequations (3.2) gives

$$\frac{\partial}{\partial t}\left(\sum_{k=-N/2}^{N/2} \hat{\eta}_k e^{ikx}\right) + i\Omega\left(-i\partial_x\right)\left(\sum_{k=-N/2}^{N/2} \hat{\eta}_k e^{ikx}\right) + \frac{3}{2}\left(\sum_{k=-N/2}^{N/2} \hat{\eta}_k e^{ikx}\right)\frac{\partial}{\partial x}\left(\sum_{u=-N/2}^{N/2} \hat{\eta}_u e^{iux}\right) = s(t). \qquad (3.3)$$

Meanwhile, from the properties of pseudodifferential operator as in [2] we obtain

$$\Omega\left(-i\partial_x\right)e^{ikx} = \Omega\left(k\right)e^{ikx}$$

Using the properties of pseudodifferentialoperator we obtain

$$\frac{\partial}{\partial t}\left(\sum_{k=-N/2}^{N/2} \hat{\eta}_k e^{ikx}\right) + i\Omega\left(k\right)\left(\sum_{k=-N/2}^{N/2} \hat{\eta}_k e^{ikx}\right) + \frac{3}{2}\left(\sum_{k=-N/2}^{N/2} \hat{\eta}_k e^{ikx}\right)\frac{\partial}{\partial x}\left(\sum_{u=-N/2}^{N/2} \hat{\eta}_u e^{iux}\right) = s(t). \qquad (3.4)$$

And finally on obtains

$$\frac{\partial\hat{\eta}}{\partial t} + i\Omega\left(k\right)\hat{\eta} + \frac{3}{2}\sum_{\substack{k=-N/2 \\ m=u+k \\ |k|\le N/2}}^{N/2} iu\left(\hat{\eta}_k\hat{\eta}_u\right) = \hat{s}(t). \qquad (3.5)$$

The coefficient $\hat{\eta}_k$ canbe found by solving equations (3.5). In equations (3.15), there is a nonlinear term which gives convolution form in spectral space. According [6], the convolution operation makes the computation more expensive. So that we need to evaluate the sum of this component by using pseudospectral evaluation. The pseudospectral evaluation is done via the following steps (see [5,6]) :

a) Transform $\hat{\eta}_k$ and $\hat{\eta}_u$ into physical spaceusing Fast Fourier Transform (FFT)

$$\eta_k = \mathcal{F}^{-1}\left(\hat{\eta}_k\right) \quad \text{dan} \quad \eta_u = \mathcal{F}^{-1}\left(\hat{\eta}_u\right),$$

b) Multiply $\hat{\eta}_k$ and $\hat{\eta}_u$

$$\eta_k \eta_u = \mathsf{F}^{-1}(\hat{\eta}_k)\mathsf{F}^{-1}(\hat{\eta}_u),$$

c) Apply the Fourier transform into $\eta_k \eta_u$ and we obtain

$$\mathsf{F}(\eta_k \eta_u) = \mathsf{F}\left(\mathsf{F}^{-1}(\hat{\eta}_k)\mathsf{F}^{-1}(\hat{\eta}_u)\right).$$

The final results now reads

$$\frac{\partial \hat{\eta}}{\partial t} + i\Omega(k)\hat{\eta} + \frac{3}{2}ik\mathsf{F}\left(\mathsf{F}^{-1}(\hat{\eta}_k)\mathsf{F}^{-1}(\hat{\eta}_u)\right) = \hat{s}(t). \tag{3.6}$$

The solution of mKdV equation can be found by applying Fourier transform into the solution of equations (3.6).The solution ofequations(3.6) can be done easily using ode45 in Matlab.

4. Numerical Solution of mKdV Equation

In this part, the numerical solution of mKdV equation will be shown. Since the given signal is a bichromatics wave so that the given signal is the following equation

$$s(t) = 2a\left[\cos((\varpi - v)t) + \cos((\varpi + v)t)\right]$$

or it can be written as

$$s(t) = 4a\cos(vt)\cos(\varpi t)$$

By choosing $v = 0.157$ and $\varpi = \pi$ (see [12])we obtain

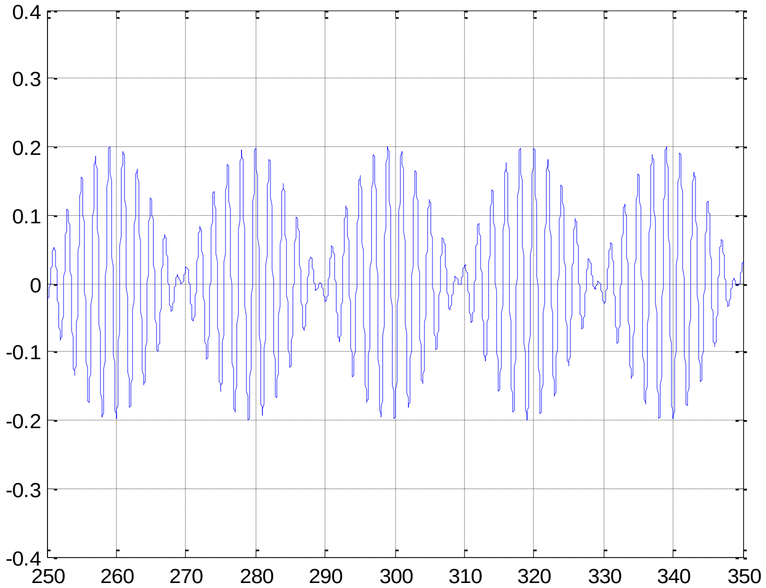

Figure 4.1 Bichromatics wave signal

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An appropriate parameter is chosen by considering the condition in towing tank. We consider a tank with dimension 5 m deep and appoximatelly 200 m long. The following are the evolution of mKdV equation with given parameters

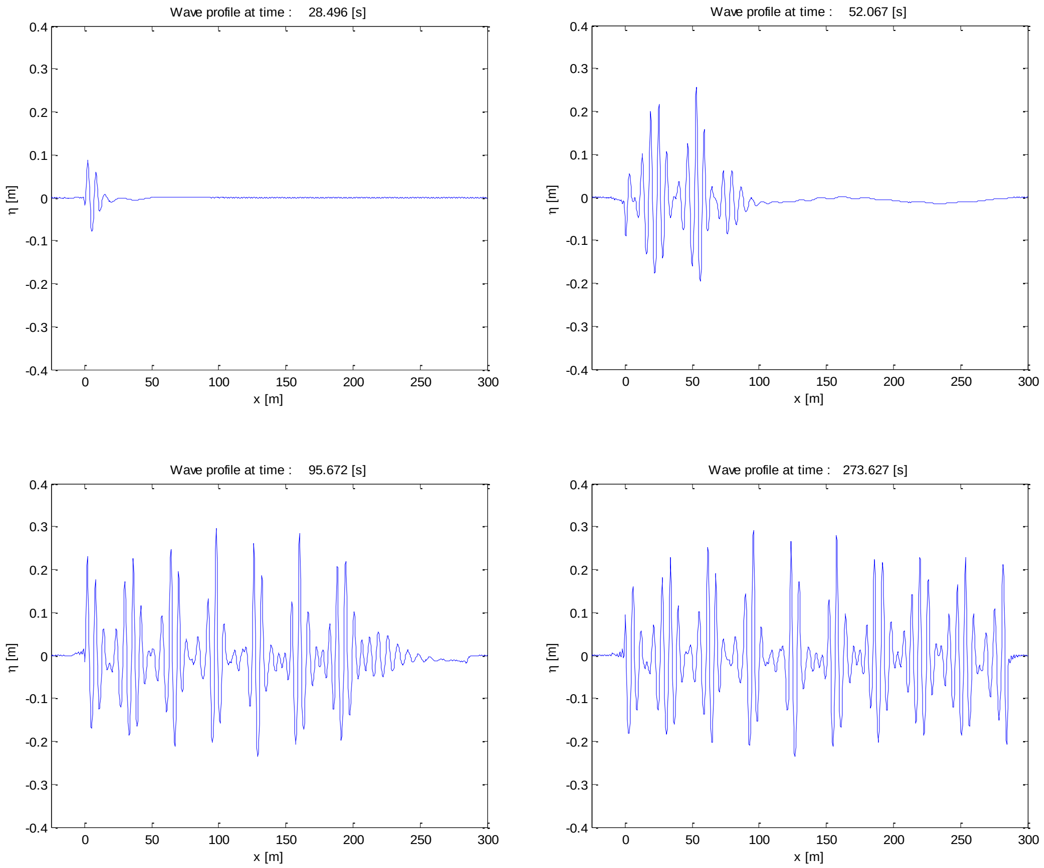

Figure 4.2 Spatial Evolution of mKdV equation

The following are the result of mKdV equation at the position $x = 10$, $x = 40$, $x = 60$, and $x = 80$

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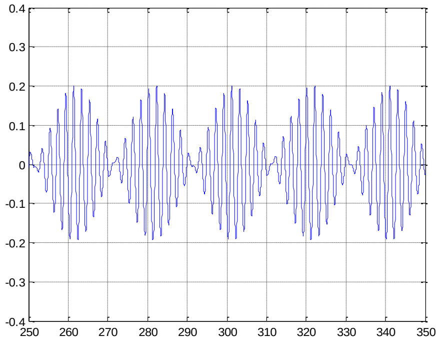
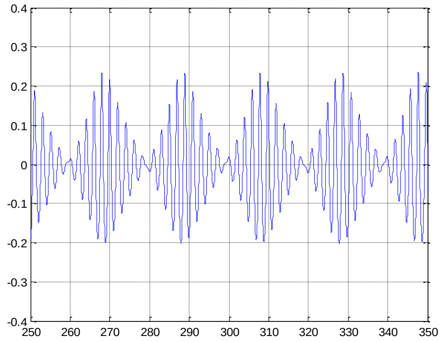
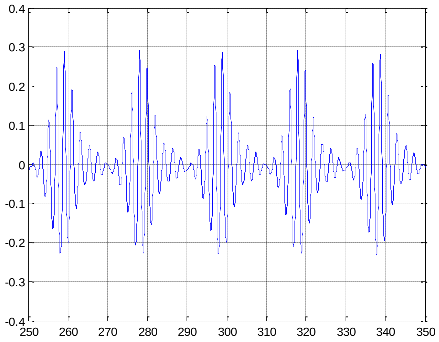


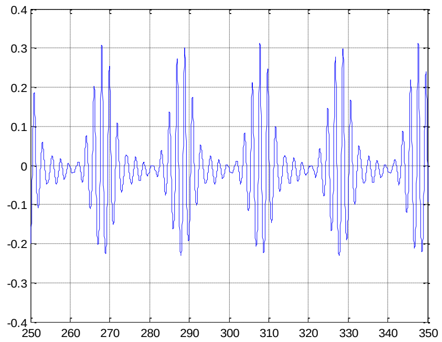
$X = 10$



$X = 40$



$X = 60$



$X = 80$

Figure 4.3 Comparison between mKdV simulation (left) and measurement data

The following we will compare the result mKdV equation with measurement data. We do not take the posisition at $x = 0$ because at this position the evolution of mKdV equation is being generating. In other words, we can not obtain a correct result at that position.

5. Comparison between mKdVsimulation and measurement data

Figure 5.1 shows the comparisonmKdV simulation and measurement data at a certain position.

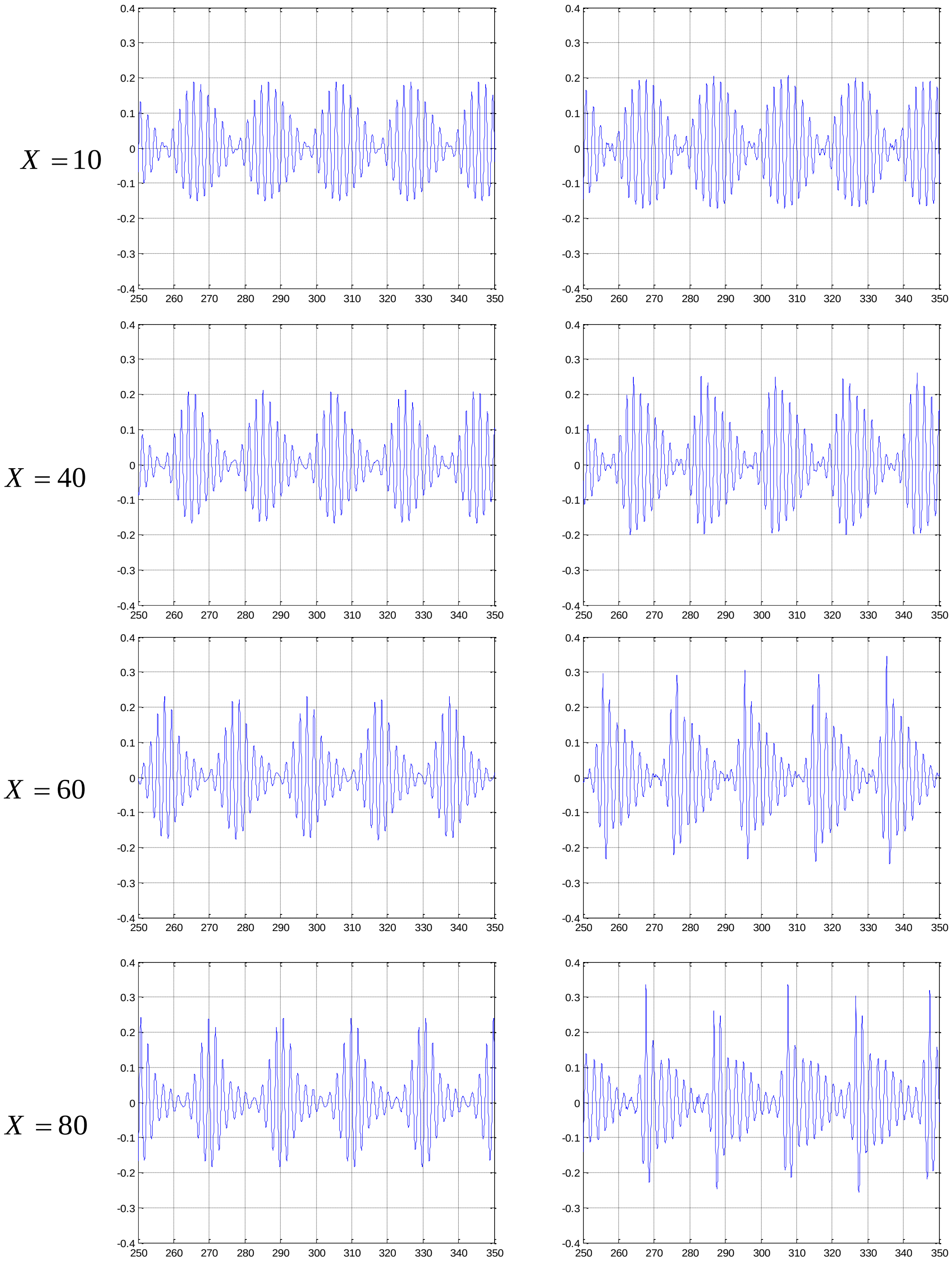


Figure 5.1 Comparison between mKdV simulation (left) and measurement data

From the figure above shows that at the position $X = 10$ the mKdV evolution gives the same profile with the measurement data. But at the position $X = 40$, $X = 60$ and $X = 80$ does not gives an exact result although the mKdV evolution still gives the same profile with the measurements data.

6. Conclusion

According to the result of the simulation of mKdV equation that mKdV equation can be solved by using pseudospectral approximation. The approximation of mKdV equation starts by approximating the solution as a Fourier series. In this approximation, the nonlinear form in mKdV equation gives a convolution sum. The problem with the convolution sum is expensive in memory usage. It can be dealt by using pseudospectral evaluation. The solution of mKdV equation can be found by using invers Fourier transform to the solution of mKdV equation at spectral space.

The comparison between mKdV simulation and measurement data shows that the solution of mKdV equation does not gives an exact height. the same height is shown at the position $X = 10$.The result at the position $X = 40$, $X = 60$ and $X = 80$ has the same profile but differ in height. A further extension of this work could be to approximate the numerical solution of KdV-type equation at uneven bottom.

7. References

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