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On Regular Elements of Semigroups of Finitary Operations

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Abstract

The set $O^n(X)$ of all n-ary operations on a set X forms a semigroup under operation $+$ defined by $f + g := f(g, , g)$. This semigroup can be considered as a particular transformation semigroup with restricted range, i.e., the semigroup of transformations with the set of all n-tuple of the same elements of X as the range. In this paper, it will be presented some results on the regularity of the elements of the semigroup from two different points of view. It will be present also the characterization of all completely regular elements and coregular elements of the semigroup as well.

Keywords: regular element, completely regular element, coregular element, semigroup, operation

1 Preliminary

Transformation semigroups belong to the most important classes of semigroups since by Cayley's theorem any semigroup is isomorphic to a transformation semigroup. Let $O^1(A)$ be the set of all transformations on A, i.e. operations with domain A and values in A. Together with the composition $\circ$ of operations defined on A one obtains the full transformation semigroup $\mathcal{O}^1(A) := (O^1(A); \circ)$. Any subsemigroup of $\mathcal{O}^1(A)$ is called a transformation semigroup. Let $O^{1-1}(A)$ be the set of all bijective transformations on A. The full transformation semigroup is regular, i.e. every element $f \in O^1(A)$ is regular in the sense that there is an operation $g \in O^1(A)$ such that $f \circ g \circ f = f$.

Let $O^n(A)$ for a fixed natural number $n \geq 1$ be the set of all n-ary operations defined on A, i.e. operations with domain A^n and values in A. The composition of $f \in O^n(A)$ with $g_1, \ldots, g_n \in O^n(A)$ is defined by

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_n) := f(g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots x_n)) \text{ for all } x_1, \ldots, x_n \in A.$$

From the $(n+1)$-ary composition operation one can derive a binary operation $+ : O^n(A) \times O^n(A) \to O^n(A)$ by

$$f + g := \pi_f \circ g,$$

which $\pi_f \in O^1(A)$ and $\pi_f(x) := f(x, \ldots, x)$ for every $x \in A$. It is easy to see that the operation $+$ is associative and as generalization of $\mathcal{O}^1(A)$ one obtains a semigroup $\mathcal{O}^n(A) := (O^n(A); +)$. This semigroup, up to isomorphism, can also be considered as a particular transformation semigroup with restricted range $T(A^n, \Delta_{A^n})$, where $\Delta_{A^n} = \{(x, \ldots, x) | x \in A\}$. In general, transformation semigroup $T(X, Y)$ is a semigroup of transformation from set X to set $Y \subseteq X$. This semigroup are already studied, for instance, as we can see in [5] and [6].

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In this paper, we use $\underline{x}$ instead of $(x_1, x_2, \ldots, x_n)$. We also use $\overline{x}$ for the n-tuple consisting of the same element $x \in A$ i.e. $\overline{x} = (x, \ldots, x)$. Then we can write $\Delta_{A^n} = \{\overline{x} | x \in A\}$. With this notation, using the composition of operations, the binary operation $+$ on $O^n(A)$ can be written as

$$(f+g)(\underline{x}) := f(\overline{g(\underline{x})}).$$

Notice that some properties of $\mathcal{O}^n(A)$ were already considered in [2], especially for $A = \{0,1\}$. For $f \in O^n(A)$ we consider its restriction to $\Delta_{A^n} : f|_{\Delta_{A^n}}$. This restriction can be mapped to a unary operation $\pi_f \in O^1(A)$ which is defined by $\pi_f(a) = b$ if and only if $f(\overline{a}) = b$. Clearly, the mapping $\Psi : \{f|_{\Delta_{A^n}} | f \in O^n(A)\} \to O^1(A)$ defined by $f|_{\Delta_{A^n}} \mapsto \pi_f$ is one-to-one. The mapping π_f is a permutation i.e., $\pi_f \in O_A^{1-1}$ iff $f(\Delta_{A^n}) = A$.

2 Regular Elements

Recall that for arbitrary semigroup S, $a \in S$ is said to be regular if we can find $x \in S$ such that $axa = a$. Moreover, if in addition $ax = xa$, then a is said to be completely regular and if $x + a + x = x$, then a is said to be coregular. An element $a \in S$ is called (m,n)-regular if there is $x \in S$ such that $a = a^m x a^n$

Let $\Sigma_A^n = \{f \in O^n(A) | f(A^n) = f(\Delta_{A^n})\}$. The we have the following property.

Lemma 2.1 *Σ_A^n is the universe of a subsemigroup of $(O^n(A); +)$.*

Proof: Let $f, g \in \Sigma_A^n$. Then, for any $\underline{x} \in A^n$, there exists an n-tuple $\overline{y} \in \Delta_{A^n}$ such that $g(\underline{x}) = g(\overline{y})$. This implies $(f+g)(\underline{x}) = f(\overline{g(\underline{x})}) = f(\overline{g(\overline{y})}) = (f+g)(\overline{y})$ and thus $f + g \in \Sigma_A^n$. ■

The following theorems show some results on regularity of elements of arbitrary transformation semigroup with restricted range $T(X,Y)$.

Theorem 2.2 [4] $\alpha \in T(X,Y)$ is regular if only if $\alpha X = \alpha Y$.

Theorem 2.3 [6] The set $\{\alpha \in T(X,Y) | \alpha X = \alpha Y\}$ is the largest regular subsemigroup of $T(X,Y)$.

As a consequence of Theorem 2.2 then we have that all elements in Σ_A^n are regular. Moreover, Theorem 2.3 implies that the set Σ_A^n is the largest regular subsemigroup of $(O^n(A); +)$. This fits also a result of Butkote [2]. Here we restate this fact and reprove in our fashion using the fact that regular transformation semigroup is regular.

Lemma 2.4 *$Reg(\mathcal{O}^n(A))$ is equal to Σ_A^n and forms a regular subsemigroup of $(O^n(A); +)$.*

Proof: First we will show that $Reg(\mathcal{O}^n(A)) = \Sigma_A^n$. Let $f \in Reg(\mathcal{O}^n(A))$. Then, there is $g \in O^n(A)$ such that $f + g + f = f$ and thus for every $\underline{x} \in A^n$, we have $f(\underline{x}) = (f+g+f)(\underline{x}) = f(g(\overline{f(\underline{x})}))$. Therefore $f(A^n) \subseteq f(\Delta_{A^n})$ and hence $f(A^n) = f(\Delta_{A^n})$. Thus $f \in \Sigma_A^n$. Conversely, let f be in Σ_A^n. Since π_f is regular, then there exists $\pi \in O^1(A)$ such that $\pi_f \circ \pi \circ \pi_f = \pi_f$. Now, let g be an arbitrary element in C_π. Since $f \in \Sigma_A^n$, for every $\underline{x} \in A^n$, there exists $y \in A$ such that $f(\underline{x}) = f(\overline{y})$. Thus, we have $(f+g+f)(\underline{x}) = f(g(\overline{f(\underline{x})})) = f(g(\overline{f(\overline{y})})) = (\pi_f \circ \pi \circ \pi_f)(y) = \pi_f(y) = f(\overline{y}) = f(\underline{x})$, i.e. f is regular. Therefore, $Reg(\mathcal{O}^n(A)) = \Sigma_A^n$ and thus by Lemma 2.1, $Reg(\mathcal{O}^n(A))$ forms a subsemigroup of $\mathcal{O}^n(A)$. Moreover since every element $g \in C_\pi$ satisfies $f+g+f = f$ and $C_\pi \cap \Sigma_A^n \neq \emptyset$, then we can choose g such that $g \in \Sigma_A^n$. Hence $Reg(\mathcal{O}^n(A))$ forms a regular subsemigroup of $(O^n(A); +)$. ■

From the proof above, we know that, whenever f is in Σ_A^n and $\pi \in O^1(A)$ satisfies $\pi_f \circ \pi \circ \pi_f = \pi_f$, then we can take an arbitrary g from C_π to get $f + g + f = f$. As an alternative, when $f \in \Sigma_A^n$, we can directly define g in the following manner. For every $z \in Imf$, we choose $y \in A$ such that $z = f(\overline{y})$ and define g with $g(\overline{z}) = y$. Thus, we have $z = f(\overline{y}) = f(\overline{g(\overline{z})}) = (f+g)(\overline{z})$. Hence, for any $\underline{x} \in A^n$, we obtain $f(\underline{x}) = (f+g)(\overline{f(\underline{x})})$, i.e. $f + g + f = f$.

If the cardinality of A is equal to m, then the number of transformations on A with rank i is equal to the cardinality of the $\mathcal{D}$-class $\mathcal{D}_i = \{\pi \in O^1(A) || Im\pi| = i\}$ and is

equal to

$$|\mathcal{D}_i| = S(m,i)\frac{m!}{(m-i)}!,$$

with $S(m,i)$ is the Stirling number of the second kind which is defined to be the number of partition of A into k non-empty subsets, i.e. is equal to

$$S(m,i) = \frac{1}{i!}\sum_{k=0}^{i}\begin{pmatrix} i \\ k \end{pmatrix}(i-k)^m(-1)^k$$

(see [3]). The number $|\mathcal{D}_i|$ can also be calculated in different way as follows.

Lemma 2.5 *Let A be arbitrary finite set of cardinality m and let $1 \leq i \leq m$ be arbitrary. The cardinality of the set $\mathcal{D}_i = \{\pi \in O^1(A) || Im\pi| = i\}$ is equal to*

$$|\mathcal{D}_i| = \sum_{\substack{n_1+\ldots+n_i=m \\ n_1,\ldots,n_i \geq 1}} \begin{pmatrix} m \\ i \end{pmatrix}\frac{m!}{n_1!n_2!\ldots n_i!}.$$

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Proof: Let $\mathcal{D}_i = \{\pi \in O^1(A) | |Im\pi| = i\}$. It is clear that there are $\binom{m}{i}$ different images of elements of $\mathcal{D}_i$. Let $I = \{a_1, \ldots, a_i\}$ be arbitrary subset of A. Now, let $\mathcal{D}_{I,n_1,\ldots,n_i}$ be the set of elements of $\mathcal{D}_i$ which images are equal to $I = \{a_1, \ldots, a_i\}$ such that the number of elements of A mapped to a_k is equal to n_k for all $k = 1, \ldots, i$, i.e. $\mathcal{D}_{I,n_1,\ldots,n_i} = \{\pi \in D_i | \pi(A) = I, |\pi^{-1}(a_k)| = n_k\}$. It is clear that $n_k \geq 1$ for all $k = 1, \ldots, i$ and moreover, $n_1 + \ldots + n_i = m$. Then, the number $|\mathcal{D}_{I,n_1,\ldots,n_i}|$ is equal to the number of permutations of multiset $\{n_1 \cdot a_1, \ldots, n_i \cdot a_i\}$ with $n_1 + \ldots + n_i = m$ and is equal to

$$\binom{m}{n_1}\binom{m-n_1}{n_2}\cdots\binom{m-(n_1+\ldots+n_{i-1})}{n_i} = \frac{m!}{n_1!n_2!\ldots n_i!}.$$

By running all possibilities for the positive numbers $n_1, \ldots, n_i$ and since there are $\binom{m}{i}$ different sets I's of cardinality i, then the number $|\mathcal{D}_i|$ is equal to

$$\binom{m}{i} \sum_{\substack{n_1+\ldots+n_i=m \\ n_1,\ldots,n_i \geq 1}} |\mathcal{D}_{I,n_1,\ldots,n_i}|,$$

i.e.

$$|\mathcal{D}_i| = \sum_{\substack{n_1+\ldots+n_i=m \\ n_1,\ldots,n_i \geq 1}} \binom{m}{i} \frac{m!}{n_1!n_2!\ldots n_i!}.$$

■

Therefore, we have the following proposition.

Proposition 2.6 *Let A be arbitrary set of cardinality m. There are precisely*

$$\sum_{i=1}^{m} \sum_{\substack{n_1+\ldots+n_i=m \\ n_1,\ldots,n_i \geq 1}} \binom{m}{i} \frac{m!}{n_1!n_2!\ldots n_i!} \cdot i^{m^n-m}.$$

regular elements in $(O^n(A);+)$.

Proof: By Lemma 2.4, the number of regular elements of $(O^n(A);+)$ is equal to the cardinality of Σ_A^n. Now, let s_i be the number of elements of Σ_n^A with the size of the images are equal to i, for arbitrary $1 \leq i \leq m$, i.e. $s_i = |\{f \in \Sigma_A^n | |Imf| = i\}|$. Let $\mathcal{D}_i = \{\pi \in O^1(A) | |Im\pi| = i\}$. Then, s_i is equal to the multiplication of $|\mathcal{D}_i|$ and the number of mapping from $A^n \setminus \Delta_{A^n}$ to any set $B \subseteq A$ of cardinality i, i.e. i^{m^n-m}, that is is equal to

$$\sum_{\substack{n_1+\ldots+n_i=m \\ n_1,\ldots,n_i \geq 1}} \binom{m}{i} \frac{m!}{n_1!n_2!\ldots n_i!} \cdot i^{m^n-m}$$

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by Lemma 2.5. Since $1 \leq i \leq m$, we have precisely

$$\sum_{i=1}^{m} \sum_{\substack{n_1+\ldots+n_i=m \\ n_1,\ldots,n_i \geq 1}} \binom{m}{i} \frac{m!}{n_1!n_2!\ldots n_i!} \cdot i^{m^n - m}$$

elements of Σ_A^n which are regular. ■

In the end of this section, we want to determine all completely regular elements and all coregular elements of $(O^n(A); +)$.

Theorem 2.7 *Let f be in $(O^n(A); +)$. The following assertions are equivalent:*

(i) *f is a completely regular element.*

(ii) *For every $\underline{x} \in A^n$, there exists $y \in A$ such that $f(\underline{x}) = (f+f)(\overline{y})$.*

(iii) *$Kerf = Ker(f+f)$.*

(iv) *$|Imf| = |\Delta_{(Imf)^n}/Kerf|$.*

(v) *$Imf = Im(f+f)$.*

(vi) *$Imf = Im(f+f)|_{\Delta_{A^n}}$.*

Proof: (ii)⇒(iii) Let (ii) be satisfied. Then, every $z \in Imf$ can be expressed as $z = (f+f)(\overline{y})$ for some $y \in A$. Therefore $Imf \subseteq Im(f+f)$ and hence $Imf = Im(f+f) = Imf|_{\Delta_{(Imf)^n}}$ since $Im(f+f) = Imf|_{\Delta_{(Imf)^n}} \subseteq Imf$. This implies $|Imf| = |\Delta_{(Imf)^n}/Kerf|$. Then, each class $[\overline{y}]_{Kerf}$ in $\Delta_{(Imf)^n}/Kerf$ contains only $\overline{y}$. Thus for every $y_1, y_2 \in Imf$, if $f(\overline{y}_1) = f(\overline{y}_2)$, then it follows that $\overline{y}_1 = \overline{y}_2$. Therefore, if $(\underline{x}_1, \underline{x}_2) \in Ker(f+f)$, i.e. $(f+f)(\underline{x}_1) = (f+f)(\underline{x}_2)$, i.e. $f(\overline{f(\underline{x}_1)}) = f(\overline{f(\underline{x}_2)})$, then it follows that $\overline{f(\underline{x}_1)} = \overline{f(\underline{x}_2)}$, i.e. if and only if $f(\underline{x}_1) = f(\underline{x}_2)$, i.e. $(\underline{x}_1, \underline{x}_2) \in Kerf$. Thus $Ker(f+f) \subseteq Kerf$ and hence $Ker(f+f) = Kerf$.

(iii)⇒(iv) Let $Ker(f+f) = Kerf$. We will show that there is a bijection mapping from Imf to $\Delta_{(Imf)^n}/Kerf$. We define a mapping $\phi : Imf \to \Delta_{(Imf)^n}/Kerf$ by $\phi(y) = [\overline{y}]_{Kerf}$ for every $y \in Imf$. Obviously ϕ is well-defined and is surjective. Let $y_1, y_2 \in Imf$ and $\phi(y_1) = \phi(y_2)$. Then, $y_1 = f(\underline{x}_1)$ and $y_2 = f(\underline{x}_2)$ for some $\underline{x}_1, \underline{x}_2 \in A^n$ and $[\overline{y}_1]_{Kerf} = \phi(y_1) = \phi(y_2) = [\overline{y}_2]_{Kerf}$. Thus $[\overline{f(\underline{x}_1)}]_{Kerf} = [\overline{f(\underline{x}_2)}]_{Kerf}$, i.e. $f(\overline{f(\underline{x}_1)}) = f(\overline{f(\underline{x}_2)})$, i.e. $(f+f)(\underline{x}_1) = (f+f)(\underline{x}_2)$, i.e. $(\underline{x}_1, \underline{x}_2) \in Ker(f+f)$. Since $Ker(f+f) = Kerf$, then $(\underline{x}_1, \underline{x}_2) \in Kerf$, i.e. $y_1 = f(\underline{x}_1) = f(\underline{x}_2) = y_2$. Thus, ϕ is injective and hence is bijective. Therefore, $|Imf| = |\Delta_{(Imf)^n}/Kerf|$ since A is finite.

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(iv)⇒(v) Let $|Imf| = |\Delta_{(Imf)^n}/Kerf|$. Then, $|Imf| = |Imf|_{\Delta_{(Imf)^n}}|$ and since $Imf|_{\Delta_{(Imf)^n}} \subseteq Imf$, then $Imf|_{\Delta_{(Imf)^n}} = Imf$. Therefore, we have $Imf = Im(f+f)$ since $Imf|_{\Delta_{(Imf)^n}} = Im(f+f)$.

(v)⇒(vi) It is clear that $Im(f+f)|_{\Delta_{A^n}} \subseteq Imf$. Let now $y \in Imf$, i.e. $y = f(\underline{x})$ for some $\underline{x} \in A^n$. By (v), there exist $\underline{x}', \underline{x}'' \in A^n$ such that $y = f(\underline{x}) = (f+f)(\underline{x}')$ and $f(\underline{x}') = (f+f)(\underline{x}'') = f(\overline{f(\underline{x}'')})$. Therefore, $y = f(\underline{x}) = (f+f)(\underline{x}') = f(\overline{f(\underline{x}')}) = f(\overline{f(\overline{f(\underline{x}'')})}) = (f+f)(\overline{f(\underline{x}'')})$. Thus, $Imf \subseteq Im(f+f)|_{\Delta_{A^n}}$ and hence $Imf = Im(f+f)|_{\Delta_{A^n}}$.

(vi)⇒(ii) By (vi), every $z \in Imf$ can be expressed as $z = (f+f)(\overline{y})$ for some $y \in A$. Thus for every $\underline{x} \in A^n$, there is $y \in A$ such that $f(\underline{x}) = (f+f)(\overline{y})$.
(i)⇒(ii) Let $f \in O^n(A)$ be completely regular. Then, there exists $g \in O^n(A)$ such that $f+g+f = f$ and $f+g = g+f$. Both equations give $f+f+g = f$. Thus for any $\underline{x} \in A^n$, we have $f(\underline{x}) = (f+f+g)(\underline{x}) = (f+f)(\overline{g(\underline{x})})$. It means that there is $y = g(\underline{x}) \in A$ such that $(f+f)(\overline{y}) = f(\underline{x})$.

By assumption, we can choose $y \in A$ such that $z = f(\underline{x}) = (f+f)(\overline{y})$ for every $\underline{x} \in f^{-1}(z)$. For this $z \in Imf$, we define $g(\overline{z}) = f(\overline{y})$ and $g(\underline{x}) = y$ for every $\underline{x} \in f^{-1}(z) \setminus \Delta_{(Imf)^n}$. It is clear that g is well-defined. Moreover, by the definition, we get $(g+f)(\underline{x}) = g(\overline{f(\underline{x})}) = f(\overline{y})$ and thus $(f+g+f)(\underline{x}) = f(\overline{(g+f)(\underline{x})}) = f(\overline{f(\overline{y})}) = (f+f)(\overline{y}) = f(\underline{x})$ for all $\underline{x} \in f^{-1}(z)$. Repeating this procedure for all $z \in Imf$, we have that, f satisfies $f+g+f = f$. Now, we will show that $(f+g)(\underline{x}) = (g+f)(\underline{x})$ for all $\underline{x} \in f^{-1}(z)$. By the definition, we have $(f+g)(\underline{x}) = f(\overline{g(\underline{x})}) = f(\overline{y}) = (g+f)(\underline{x})$ for every $\underline{x} \in f^{-1}(z) \setminus \Delta_{(Imf)^n}$. If $\underline{x} \in \Delta_{(Imf)^n}$, i.e. $\underline{x} = \overline{f(\underline{x}')}$ for some $\underline{x}' \in A^n$, then $(f+g)(\underline{x}) = (f+g)(\overline{f(\underline{x}')}) = (f+g+f)(\underline{x}') = f(\underline{x}')$. On the other hand side, $(g+f)(\underline{x}) = g(\overline{f(\underline{x})}) = g(\overline{f(\overline{f(\underline{x}')})})$. Thus $(g+f)(\underline{x}) = f(\overline{y})$ for some $\overline{y}$ satisfying $(f+f)(\underline{x}') = (f+f)(\overline{y})$. Since $Ker(f+f) = Kerf$, then $f(\underline{x}') = f(\overline{y})$. Thus $(f+g)(\underline{x}) = f(\underline{x}') = f(\overline{y}) = (g+f)(\underline{x})$. Doing the procedure for all $z \in Imf$, we have $f+g = g+f$. Hence, f is completely regular. ∎

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It is clear that all completely regular elements are contained in $Reg(O^n(A)) = \Sigma_A^n$. But conversely, not all elements of $Reg(O^n(A))$ are completely regular. As an example, take the binary operation f on $\{1,2,3\}$ with $f(1,1) = f(1,2) = f(1,3) = f(2,1) = 3$ and $f(2,2) = f(2,3) = f(3,1) = f(3,2) = f(3,3) = 2$. It is clear that $f \in \Sigma_A^n$ and thus f is regular. But by simple calculation we have that $f + f = c_2^2$, i.e. $f + f$ is a binary constant operation with value 2. Thus $Ker(f + f) \neq Ker f$ and hence f is not completely regular element. Therefore, $(Reg(O^n(A)); +)$ is not a completely regular semigroup.

Now, we give a characterization on coregular elements of $(O^n(A); +)$.

Theorem 2.8 *Let f be in $(O^n(A); +)$. The following assertions are equivalent:*

(i) *f is a coregular element of $(O^n(A); +)$.*

(ii) *For every $\underline{x} \in A^n$, there exists $y \in A$ such that $f(\underline{x}) = (f + f)(\overline{y})$ and $(f + f)(\underline{x}) = f(\overline{y})$.*

(iii) $f + f + f = f$.

Proof: (i)⇒(ii) Let $f \in (O^n(A); +)$ be coregular. Then, we can find $g \in O^n(A)$ such that $f + g + f = f = g + f + g$. Substituting $f = g + f + g$ to the first f and to the second f of $f + f$ respectively, we have $f + f = (g + f + g) + f = g + f + g + f = g + (f + g + f) = g + f$ and $f + f = f + (g + f + g) = f + g + f + g = (f + g + f) + g = f + g$. Therefore $g + f = f + f = f + g$ and hence $f + f + g = f + g + f = f$. Thus for every $\underline{x} \in A^n$, we have $f(\underline{x}) = (f + f + g)(\underline{x}) = (f + f)(\overline{g(\underline{x})})$. Hence, we can find $y = g(\underline{x}) \in A$ such that $f(\underline{x}) = (f + f)(\overline{y})$. Moreover, by $f + g = f + f$, then this $y = g(\underline{x})$ also satisfies $(f + f)(\underline{x}) = (f + g)(\underline{x}) = f(\overline{g(\underline{x})}) = f(\overline{y})$.
(ii)⇒(iii) Let (ii) be satisfied, i.e. for every $\underline{x} \in A^n$, there exists $y \in A$ such that $f(\underline{x}) = (f + f)(\overline{y})$ and $(f + f)(\underline{x}) = f(\overline{y})$. Then, we have $f(\underline{x}) = (f + f)(\overline{y}) = f(\overline{f(\overline{y})}) = f(\overline{(f + f)(\underline{x})}) = (f + f + f)(\underline{x})$. Therefore $f = f + f + f$.
(iii)⇒(i) If $f + f + f = f$, then it is clear that f is coregular element. ■

Remark 2.9 The fact that an element a of a semigroup is coregular if and only if $a^3 = a$ was already shown in [1].

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