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A Pedagogical Value From Mathematical Mistakes

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Abstract

A real correct and mistake basically have the same point. Mistake done by student during doing mathematics problem can have the positive values and worth to develop the new mathematics concept. In mathematics teaching, process to understand a concept sometimes will be easier if besides gave a lead real correct and also accompanied by the counter of example from looking the aspect of mathematics mistake good for student and to learn in developing values pedagogic such thinking critical, logical, analytical in exploring to born the new mathematics. We often dismiss a mistake as something to be avoided and at best to be corrected as soon as possible. Many types of algebraic mistakes and purely arithmetic ones and discusses how they can be channeled into positive, useful learning, and growing experience. Mathematical Mistake can be used as a springboard for developing new mathematics and can be use to turn negative experiences into positive ones.
Key Words*:* Mathematical Mistake, Pedagogical Values

I. INTRODUCTION

1. Background

We often glide through a lesson with satisfaction as long as our students are feeding back what we want them to learn. When they make a mistake we simply correct it and go ahead. It's rarely do we capitalize on a mistake as an incident of high potential. When an obviously incorrect procedure yields a correct result, we may feel puzzled and curious to know how and why this could have happened. Trying to answer these questions can involve us not only problem solving but in problem-posing activities as well. Many teachers would automatically categorize a mistake as an evaluative experience for both the teacher and the student, but this not always a necessary avenue to follow. This experience can provide the opportunity for creativity even in mathematics basic level.

Mathematical Mistakes from the Classroom

Crouse and Sloyer (1978) reveal that, after several years of teaching a method course for college student preparing for secondary mathematics teaching, they found that these student teachers had great difficulty answering mathematical questions raised by their students. The majority of questions come from actual classroom experience.

 ISBN : 978 – 979 – 16353 – 7 – 0

Some example of mathematical mistake during instruction in class found by Crouse and Sloyer can be showed as follows.

1. You put the problem $\frac{1}{5}+\frac{2}{3}$ on the board and ask one of your students to come to the board and demonstrate how to compute the answer. He writes the following: $\frac{1}{5}+\frac{2}{3}=\frac{3}{8}$. You tell him that is wrong, but he gives the explanation, "if a baseball players are up at bat 5 times on the first day and gets one hit, and in the second day he gets 2 hits out of 3 times at bat, then altogether he has 3 hits out of 8 times at bat. "How do we reply?
2. A student hands in the following solution when asked to reduce $\frac{16}{64}$ to Lowes terms. $\frac{1\cancel{6}}{\cancel{6}4}=\frac{1.1}{1.4}=\frac{1}{4}=\frac{1}{4}$. Is the student correct?
3. A student hands in the following work.
$\frac{x+6}{2}=x+\frac{6}{2}=x+3$. When the student asks us why he can't the 2 into 6, how would you answer him?
4. A ten grade student hands in the following work.
$\frac{x^{\cancel{2}}-y^{\cancel{2}}}{x+y}=x-y$. The student wants to know why the method is wrong. How would you answer?
5. You ask the students to simplify $\frac{x^2-9}{x-3}$. One student does the following:
$\frac{x^2-9}{x-3}=\frac{x^2-9}{x-3}=\frac{x-9}{-3}=\frac{x+\cancel{9}^{3}}{\cancel{3}}=x+3$
Is the student correct? Will his method work all the time?
6. You ask the students to simplify $\frac{1}{x}+\frac{1}{x}$. A student writes $\frac{1}{x}+\frac{1}{x}=2$. What is the student's problem?
7. A student says that 3a – 2a = 1. How would you help him?
8. A student is evaluating the expression $\frac{3^2+7^2}{\left(\frac{58}{10}\right)^2}$

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and the student does the following: $\dfrac{3^{\cancel{2}} + 7^{\cancel{2}}}{\left(\dfrac{58}{10}\right)^{\cancel{2}}} = \dfrac{10}{\left(\dfrac{58}{10}\right)} = \dfrac{100}{58}$. Which is the correct answer? How would you help this students?

9. A student hands in the following work for the following problem.

   Solve:

   $x^2 – 14x + 24 = 3$
   $(x – 12)(x – 2) = 3$
   $(x – 12)(x – 2) = 3 . 1$
   $x – 12 = 3$ or $x - 2 = 1$
   $x = 15$, $x = 3$
   $x \, \varepsilon \, (3, 15)$. Is the student correct?

10. A Student hands in the following solution: $(a + b)^2 = a^2 + b^2$ because, except for the operation being different, this problem is just like $(a . b)^2$, which is equal to $a^2 . b^2$. Therefore, it follows that $(a + b)^2 = a^2 + b^2$. How would you help this student?

II. DISSCUSSION AND ANALYSIS

The following is a possible treatment of a classical mathematical mistake. The student square $(a + b)$ to get $a^2 + b^2$, leaving out the 2ab term.

1. The teacher may ask the student to substitute a numerical example; comparing that result with normal multiplication will hopefully confine the student that
   $(a + b)^2 \neq a^2 + b^2$
2. The student may show correct method of squaring $(a + b)$, algebraically or geometrically, thus introducing the "middle term" as something demonstrated by proof. The first option uses falsifiability to persuade the student; the second provides "positive" proof.

This system is showing the student that his belief is wrong and then countering with the correct method make two basic assumptions:

(i) the student's belief in his original 'mistake' was not strong;

(ii) the error was roughly speaking 'random'; that is, the pupil did not base his conclusion on any mathematical or psychological idea previously explored by the teacher or the student.

There are cases where these assumption do hold, but is substituting numerical

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examples enough to dispel a *very strong* belief that $(a + b)^2 = a^2 + b^2$, and more importantly, what mathematical or psychological reasons did the student have for arriving at his conclusion? The belief as symptom of what may be a serious disease or even more than one disease by substituting numerical examples, or by providing proof, the teacher has deal with the symptom only and left the disease to reach epidemic proportion.

The following assignment was given to student in a method of teaching mathematics' course by Professor S.I Brown at State University of NY. At buffalo.

(i) Justify each of the following classical mathematics mistake in as many ways as possible, *given* that you belief each correct.

A. $(a + b)^2 = a^2 + b^2$

B. $a/b + c/d = (a + c)/(b + d)$

C. $\varnothing = \{\varnothing\}$

(ii) Re-examine the justifications in part one and discuss what you believe to be the reason (mathematical or psychological) behind the mistakes.

(iii) Make those mathematical mistakes a positive experience for the students.

Some of the following justifications are a direct result of the work done on this assignment.

Part A. $(a + b)^2 = a^2 + b^2$

(i) The distributive law of squaring over addition.

(ii) Induction: $(a + b)^1 = (a^1 + b^1) \Rightarrow (a + b)^2 = a^2 + b^2$.

(iii) Consider the Pythagorean theorem, $a^2 + b^2 = c^2$. If $c = a + b$, then $a^2 + b^2 = (a + b)^2$.

(iv) Say the following sentence fast. The sum of the squares equals the square of the sum. The attraction this has is that is *sounds* correct.

(v) Since $(ab)^2 = a^2b^2$ and since multiplication is just a quick form of addition, then…

(vi) How many times have we as teachers told our pupils, "Whatever we do to the left side of an equation we must also do to the right side?" Start with the following:

$(a + b)(a – b) = a^2 – b^2$.

Now, apply the principle above by changing the negative sign in the left side of the equation to a plus, and changing the negative sign on the right side to a plus. Then $(a + b)(a + b) = a^2 + b^2$.

All six of these explanations have two things in common. One is that each has

 ISBN : 978 – 979 – 16353 – 7 – 0

used some principle learned previously and the other is each has the compelling notion that "in mathematics, the future must look like the past". (Special acknowledgement for this observation goes to Prof. S. I. Brown.)

Part B. (a/b) + (c/d) = (a + c)/(b + d)

(i) This looks like multiplication:
(a/b)(c/d) = (ac)/(bd).
and since multiplication is really a fast way of doing addition…

(ii) The "baseball analogy" was first proposed by Dorothy Buerk (Brown & Walter, 1993) who assisted, along with others, in the teaching of the methods course. Anyone who follows baseball knows that if a batter has 3 hits out of 5 attempts on Monday and has 2 hits out of 3 attempts on Tuesday, his combined record is 5 hits out of 8 attempts. Therefore it follows that (3/5) + (2/3) = 5/8, or in general terms.

Part C. ∅ = {∅}

(i) The null set is nothing and the set of nothing is nothing, therefore 0 = {0}.

(ii) Notation argument: All other sets use brackets therefore we must use brackets in this case.

(iii) One student claimed that 0 = {0} = { }, but {0} is best because it leaves "*no doubt*" that we are talking about the empty set. Redundancy stresses the point.

Again, in examples B and C there is a strong needed to make the future looks like the past. The multiple reasons given for each mistake suggest that a simple correction by the teacher is insufficient. For example, if a student give induction as his reason that $(a + b)^2 = a^2 + b^2$. Justification (vi) offers yet another disease, the pupils' tendency to generalize rules that are used rather loosely in the classroom. As you can see in this example the rule was applied rather than ingeniously.

Examples B and C offer rich ground for determining and perhaps beginning to diagnose mathematical diseases. Perhaps a list of questions will help us begin to re-examine what may be at the root of the mistakes by Mayerson (Brown & Walter, 1993: 153).

1. Where are these justifications in the notion of variable misunderstood?
2. The baseball analogy demands that we take a closer look at so-called everyday occurrences and attempt to see what they may imply mathematically.

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3. How confusing is notation?
4. How much do we as teachers underestimate the ability of our students to derive new mathematical principles, e.g. the 'no doubt' principle?
5. Some of the justifications of 0 = {0} imply that there is a confusion between the meanings of 'is an element of' and 'is a subset of'. How does this misconception relate to other misconceptions students have in beginning set theory?

Other Mathematics mistake by Borasi (Brown & Walter, 1993: 159) "Algebraic Explorations of the Error $\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$ "

Why does such an absurd simplification produce the correct result? Is this example the only case for which this kind of simplification work? We can attempt to answer both questions at one time, by starting the more general problem: For what values of the digits *a, b,* and *c,* is

$\frac{10a+b}{10b+c} = \frac{a}{c}$? Or, equivalently, what are the integral solution between 1 and 9 of the following equation?

(1) $(10a + b)c - a(10b + c) = 0$

The values $(a, b, c) = (1, 6, 4)$ satisfy this equation, which explains why the result of the simplification turned out to be correct in the specific case presented. Do other solutions exist? How can we search for them? We do not have a straight-forward algorithm that can be applied to solve equations of this kind, but we can try several approaches.

For example, we can try to rewrite equation (1) in different ways to see if anything may be revealed. For example:

(2) $10a(b - c) = c(b - a)$

(3) $10ab = c(9a + b)$

(4) $9ac = b(10a - c)$

Equation (2) may presents some advantages, as all a, $|b - c|$, c, and $|b - a|$ must be less than 10. We can then observe that since 5 divides the first side and 5 is a prime number, either c = 5 or $|b - a|$ = 5. In the example, we had, in fact, b – a = 6 – 1 = 5. We can now see if c = 5 in some solutions. With this extra condition, equation (2) becomes

 ISBN : 978 – 979 – 16353 – 7 – 0

$10a(b - 5) = 5(b - a)$ or

(5) $b = \dfrac{9a}{2a - 1}$

Computing from (5) the values of b corresponding to a = 1, 2, …, 9, we *do* find two new solutions besides trivial one:

$(a, b, c) = (1, 9, 5) \quad \Leftrightarrow \quad \dfrac{1\cancel{9}}{\cancel{9}5} = \dfrac{1}{5}$

$(a, b, c) = (2, 6, 5) \quad \Leftrightarrow \quad \dfrac{2\cancel{6}}{\cancel{6}5} = \dfrac{2}{5}$

We have found all the possible solutions with c = 5. If other solutions exist, they must derive from $|b - a| = 5$, that is, when either b = a + 5 or a = b + 5. At first sight checking this case may seem more complicated than checking c = 5, but it is actually less so. For b = a + 5, equation (2) becomes

$10a(a + 5 - c) = 5c$, or

(6) $c = \dfrac{2a^2 + 10a}{1 + 2a}$

And this time we have only to check for a = 1, 2, 3, 4 in (6), as it must be that b = a + 5 < 10. We thus find two nontrivial solutions, one of which is our original one:

$(a, b, c) = (1, 6, 4) \quad \Leftrightarrow \quad \dfrac{1\cancel{6}}{\cancel{6}4} = \dfrac{1}{4}$

$(a, b, c) = (4, 8, 9) \quad \Leftrightarrow \quad \dfrac{4\cancel{9}}{\cancel{9}8} = \dfrac{4}{8}$

In the case of a = b + 5, equation (2) becomes

$10(b + 5)(b - c) = -5c$, or

(7) $c = \dfrac{2b^2 + 10b}{9 + 2b}$

Checking for b = 1, 2, 3, 4 (again it must be that a = b + 5 < 10) in (7), we find no other solution. This situation may become a rich source of new problems once and we challenge the way that we have stated the problem previously (equation 1) or modify some of its elements. For example, we assumed that the numbers were written in the usual decimal notation. What if the base of numeration was not ten but another natural number k? The problem would then be to find the integral solutions between 1 and (k – 1) of the equation: $c(ka + b) - a(kb + c) = 0$.

 ISBN : 978 – 979 – 16353 – 7 – 0

It may be interesting to discuss the values of k to which we can still apply the argument used in this paper.

In this article, we have also limited our consideration to two and one digit numbers. Can we come up with analogous “simplifications” using more digits? For example, what about $\frac{5\cancel{9}4}{2\cancel{9}7} = \frac{54}{27}$? Finding all “three-digit fractions” that can correctly be simplified in this way will now involve a lot more cases. Even if we use a computer, we will face the real challenge in writing an efficient program and eliminating a priori as many trivial solutions as possible (you can expect hundreds of solutions in this case!). What are other possible simplifications that can occur with “three-digit fractions”? What is the percentage of “correct “versus “wrong” result of each simplification? Does any pattern occur in the solutions?

This problem can provide concrete material and the stimulus for a discussion about the difference between necessary and sufficient conditions for solutions and about the values and limitations of heuristic procedures versus algorithms in solving equations. It can also provide further reflection on the use of computers in mathematics, in comparison to more “classical” mathematical activities.

There are many questions and strategies to be explored; these were just a few. We could do well to re-examine our pupils’ mistakes and initiate investigations that will no doubt lead to new pedagogical experiences for ourselves and positive learning experiences for both teachers and pupils.

Pedagogical Values Expanding

Mathematical mistake has the education values and can be used;

1. To develop new mathematics and to turn a negative experience (making an error) into a positive experience.
2. Instead of showing pupils that their mistake is wrong, ask them when it is right; that is, what questions can we ask to make the solution correct? Another example of this type of activity can be seen by exploring mistake B where the pupils add the numerators and the denominators instead of finding a common denominator. Consider the solution (a + c) / (b + d). In addition to reviewing addition of fractions one can ask, when can we add the numerators and

 ISBN : 978 – 979 – 16353 – 7 – 0

denominators together in such a fashion? Pupils can now explore questions such as:

(i) How many ways are there to find a fraction between two other fractions on a number line?

(ii) Is (a + c)/(b + d) always between a/b and c/d?

(iii) Can we talk about between's in two or higher dimensions?

3. If the solution is correct, how does this affect the rest of mathematics? And similarly what changes have to be made in our system to accommodate our solution? For example, let us consider again the error, $(a + b)^2 = a^2 + b^2$. How can we change the system to accommodate this or similar expressions? One consequence is that when two *different* numbers are multiplied together their product is zero. This eliminates the 2ab term in $(a + b)^2$. How does this change affect the rest of our system? What happens to $(a + b)^3$, $(a + b)^4$, etc.?
4. Are there other existing mathematical systems in which our solution holds? A classic example can be found in the field of geometry. There are many constructions that cannot be done by Euclidean methods and can be done by non-Euclidean methods.

Pursuant to finding and analyze to mathematical mistake, as we have been studied above, hence values which can be developed shall be as follows:

1. *Having knowledge*. Through new knowledge mathematical mistake in outside will give a new information too. Skills of gathering the information involve such ability such as: reading, counting/calculating, and doing observation in where the student learn to calculate non merely learning mechanistic to apply the number calculation but interpreting what information obtained from that calculation. Contextualization Mathematics become of vital importance for student to understand the meaning from what they have learned in their daily life as information.
2. *Using scientific skills and scientific thinking.* Think by scientific skills and scientific thinking are core from new knowledge invention, but the process of invention is not walking smoothly, but a lot of met the failure and mistake. Therefore mathematical mistake can becomes to access potential to find the new very amazing mathematics.

 ISBN : 978 – 979 – 16353 – 7 – 0

3. *Thinking strategically*. To follow up the mistake, we should try the investigation of important features from the mistake needs to think strategically. In this case, values which can be taken away from a mathematics mistake is try to make a mapping of the cognitive or metacognitive and apply it in problem solving of mathematical mistake.
4. *Long life learning skills*. Learn through mistake is one of principle from long life learning. With this view learn the mathematics will be wider because study do not only through example just real correct case but also can be developed from mathematics mistake. Mathematics mistake represent the starting points to learn to investigate further.
5. *Communication skill, Skill of research and exploration, and able to use technology to further developing their scientific.* Mathematics' mistake done by a student need the explanation or communication. The Mistake possibility resulted from by a perception mistake to problem which is all student face. In this case, possible they see the other side at the opposite of mathematics problem which we have given to them. Try the understanding way of thinking; the students are given the opportunity to find the new mathematics which is unforeseen possible previously. This invention, sometimes require to be made more common, so that we require the technological skill like computer program application where use computer application will quicken the invention process with the more gratifying result.
6. *Critical, creative and independent thinking.* Resolving to mathematical mistake needs the ability to think critically, creative and independent thinking. Only one who own the opinion of like this which can see the positive values from mathematical mistake and also able to think that people can find something from the outside of the mistake. Therefore the way of thinking requires to be developed at school and then apply it in the form of problem base instruction.
7. *Decision making*. If knowledge has been processed to become a conclusion, hence people can take the decision pursuant to the conclusion. This fact indicates that the people oftentimes fear to take the decision for fear of facing the mistake which emerge. But, if mistake is the basic to make decision hence somebody will be getting self confidence in his/her life.

8. *Problem solving and Problem Posing.* Problem Solving is the heart of mathematics, basically there no mathematics without problem solving. In this case, mathematics' mistake is potential source of problem to make somebody becomes the problem solver. Problem solving needs the creativity and wisdom such creativity to find the efficient and effective to find solving, while wisdom is needed because resolving have to always pay attention to the relevancy environmentally. Therefore early on student require learning to solve problem as according to level thinking of it (Brown & Walter, 2005).

III. CONCLUSION AND SUGGESTIONS

1. Conclusion

Mathematical mistakes that either students or teachers faced are not cases to be avoided, but how we learn more from it to consider further then can perform a new mathematics. And to realize it, let's consider some smart steps to change making an error to be a good experience in mathematics learning and teaching process. Based on analyzing that mathematical mistake, can be developed new mathematics and to turn a negative experience (making an error) back into a positive experience.

2. Suggestions

Some pedagogical values which can be developed is life skills value as following:

a. *Having knowledge*. Some steps can be used in gathering information from the outside of the life to get new information such reading, counting, and observation in where the student learn to calculate non merely learning mechanistic to apply the number calculation, but interpreting what information obtained from that calculation
b. *Using scientific skills and scientific thinking.* Think by scientific skills and scientific thinking is core from new knowledge invention, but the process of invention is not walking smoothly, but a lot of met the failure and mistake. Therefore mathematical mistake may as access potential to find the new very amazing mathematics.
c. *Thinking strategically*. In this case, values which can be taken away from a mathematics mistake is try to make a mapping of the cognitive or metacognitive and apply it in problem solving of mathematical mistake.

d. *Long life learning skills.* Learn through mistake is one of the principle of long life learning. With this view learn mathematics will be wider because study does not only through example, but also can be developed from mathematics mistake. Mathematics mistake is a representation of starting points to learn and to investigate further more.

e. *Communication skill, Skill of research and exploration,* from the mathematics mistake, the students must be able to explore the case they faced, communicate it and looking it back is possible for them to see the other side at the opposite of mathematics problem which the teacher has given to them.

f. *Critical, creative and independent thinking.* Resolving to mathematical mistake need the ability to think critically, creative and independent thinking. and also able to think that people can find something from the outside of the mistake.

g. *Decision making.* The students can take the decision pursuant to the conclusion. This case indicates that the people are often feared to take decision in facing the mistake which is emerging. Further more, the mistake in making decision hence somebody in getting self confidence in his/her life.

h. *Problem solving and Problem Posing.* Everyone is potentially may becomes a problem solver. Therefore early on student require learning to solve problem is according to the level of thinking it. Problem solving needs more creativity to find the efficient and effective to find solving and always pay attention to the relevancy environmentally.

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