ON THE GEOMETRIC QUANTIZATION OF POISSON MANIFOLDS

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Abstract

The geometric quantization on Poisson manifolds is discussed in the spirit of the fact that the notion of Poisson manifolds is the dual generalization of the notion of symplectic manifolds as phase spaces of classical mechanics. A possible notion of polarization constructed from Pfaffian system on Poisson manifolds extending the usual notion of polarization on symplectic manifolds was suggested. Some examples of Poisson manifolds and their quantization are presented.

Abstract

(in Indonesian)

Telah dibahas pengkuantuman geometrik pada manifold-manifold Poisson berangkat dari kenyataan bahwa konsep manifold Poisson merupakan perumuman yang bersifat dual dari konsep manifold simplektik sebagai ruang fase dalam mekanika klasik. Suatu konsep polarisasi yang disusun dari sistem Pfaffian pada manifold-manifold Poisson diusulkan sebagai perluasan atas konsep polarisasi biasa pada manifold simplektik. Disajikan beberapa contoh manifold Poisson dan pengkuantumannya.

1 Introduction

One of the major problems in mathematical physics is to find an appropriate mathematical language for quantization procedure. It contains some translation instructions which translate observables and states of a classical -macroscopic- system into observables and states of a -microscopic- system obeying quantum mechanical laws. The mathematical model for states and observables of a classical system is usually a symplectic manifold, i.g. a pair (M, ω) consisting of a differentiable manifold M and a closed non-degenerate 2-form ω on M called *symplectic structure*, and the set $C^\infty(M, \mathbb{R})$ of smooth real valued functions f on M as generators of a subgroup of symplectic (canonical) diffeomorphisms of M via Hamiltonian vector fields X_f satisfying

$$i_{X_f}\omega + df = 0. \tag{1}$$

The existence and uniqueness of the solution of Eq.(1) is guaranted by the non-degeneracy of ω. The symplectic structure ω in turn induces Poisson bracket $\{\cdot,\cdot\}_\omega$ on $C^\infty(M, \mathbb{R})$ according to

$$\{f, g\}_\omega = \omega(X_g, X_f), \qquad \forall f, g \in C^\infty(M, \mathbb{R}) \tag{2}$$

so that the pair $(C^\infty(M, \mathbb{R}), \{\cdot,\cdot\}_\omega)$ is a Lie algebra called *Poisson algebra*. Associated with the symplectic structure ω there exists uniquely a bivector Λ of rank $dim(M)$ everywhere on M satisfying $\omega(X_g, X_f) = \Lambda(df, dg)$, for every $f, g \in C^\infty(M, \mathbb{R})$. On the other hand, the mathematical model for states and observables[1,2] of quantum mechanics is a complex separable Hilbert space $\mathcal{H}$ and the set $Sym(\mathcal{H})$ of symmetric operators on $\mathcal{H}$ generating a group of unitary operators.

Dirac is the first who tried to establish a general quantization scheme, which usually is known as canonical quantization. Though Dirac's idea is very convincing and fruithful from the point of view of physics, it is somewhat lacking of mathematical rigor : it is applicable only on the systems with finite dimensional flat classical phase spaces, it needs the existence of global canonical coordinates, it is not invariant under canonical transformations, and serious difficulties arise when we deal with constrained systems or with internal degree of freedom. The theory of geometric quantization dates back to the work of Kostant ([9]), Souriau ([15]) and Kirrilov ([8]). The main goal of geometric quantization is to set a relation between classical and quantum mechanics from geometrical point of view, taking canonical quantization as model. In other word, geometric quantization attempts to generalize canonical quantization and remove the above mentioned problems surrounding canonical quantization. For instance, it generalizes canonical quantization for classical phase spaces which are not necessarily flat and, even without being a cotangent bundle, namely symplectic manifolds. We will recapitulate the theory to whatever extent necessary in Section 2.

There are at least two directions in which one generalizes symplectic manifolds in natural way as classical phase spaces. In the first direction, one removes the non-degeneracy condition on ω. This relaxation leads to a more general notion of phase

[1] In the usual (or sharp) quantum mechanics one makes use of self-adjoint operators to represent observables. But, in order to accomodate the unsharp quantum mechanics ([3, 12]), we take rather the set $Sym(\mathcal{H})$ which contains also all self-adjoint operators on $\mathcal{H}$.

[2] There are more rigorous objects modeling the states and observables of quantum mechanics ([12]). However, for our needs here $\mathcal{H}$ and $Sym(\mathcal{H})$ are already sufficient.

spaces which are refered as to *presymplectic manifolds.* The weaker condition on ω implies clearly the following fact : for an element $f \in C^\infty(M, \mathbb{R})$ Eq.(1) may or may not have a solution. When it has, the solution is not unique. The generalization is useful, for instand, when the physical system under consideration is subjected to certain constraints ([6, 1]). A general approach of geometric quantization on presymplectic manifolds is discussed in [16]. The second direction is in some sense dual to the first one, in which one appreciates the bivector Λ in its generality instead of the symplectic structure itself. The generalization is closed related to quantum groups and deformation quantizations [2, 4]. A *Poisson manifold* is a pair (M, Λ) consisting of a smooth manifold M and a bivector Λ on M (not necessarily of rank $dim(M)$ everywhere) so that $[\Lambda, \Lambda]_{SN}$ vanishes, where $[\cdot,\cdot]_{SN}$ is a map $[\cdot,\cdot]_{SN} : \mathfrak{X}_p(M) \times \mathfrak{X}_q(M) \to \mathfrak{X}_{p+q-1}(M)$ called Schouten-Nijenhuis bracket ([17]) and $\mathfrak{X}_{p'}(M)$ is the set of all p'-vector fields on M.

2 Geometric quantization on symplectic manifolds

We revisite shortly geometric quantization on symplectic manifolds. The detail account can be found in [9, 14, 15, 18]. We begin with a general definition of quantization under which the geometric quantization is subordinated :

Definition 2.1 *Let $(C^\infty(M, \mathbb{R}), \{\cdot,\cdot\}_\omega)$ be a Poisson algebra of a symplectic manifold (M, ω) and $\mathcal{H}$ be a separable Hilbert space. A linear mapping $\mathcal{Q} : C^\infty(M, \mathbb{R}) \to Sym(\mathcal{H})$ from the Poisson algebra $C^\infty(M, \mathbb{R})$ into the set $Sym(\mathcal{H})$ of all symmetric operators in $\mathcal{H}$ is called prequantization if it satisfies the following axioms :* **(a)** *$\mathcal{Q}(c) = c\mathbb{I}$, where $\mathbb{I}$ denotes the identity operator in $\mathcal{H}$ and $c \in \mathbb{R}$, and* **(b)** *$\mathcal{Q}(\{f, g\}) = i[\mathcal{Q}(f), \mathcal{Q}(g)]$ for every $f, g \in C^\infty(M, \mathbb{R})$, where $[\cdot,\cdot]$ is commutator in $\mathcal{H}$. Whenever $\mathcal{Q}$ is also an irreducible representation of $C^\infty(M, \mathbb{R})$ in $\mathcal{H}$, we call it quantization.*

However, such irreducible map does not exist in general, even for the case of $\mathbb{R}^2$ with the canonical Poisson structure ([7, 5]).

Now the first step to geometric quantization is Kostant-Souriau (KS) prequantization. For a symplectic manifold (M, ω), KS-prequantization consists of choosing a *prequantization line bundle*, i.e. complex line bundle $(L, \bigtriangledown, < \cdot,\cdot >)$ with hermitian connection so that its curvature $Cur(L, \bigtriangledown)$ in L is equal to ω. However, not all symplectic manifold admits such line bundle. A symplectic manifold is called *prequantizable* if the manifold admits prequantization line bundle. The following theorem gives us the characterization of the prequantizability of a symplectic manifold.

Theorem 2.1 *There exists a complex line bundle $(L, \bigtriangledown, < \cdot,\cdot >)$ with hermitian connection over (M, ω) so that $Cur(L, \bigtriangledown) = \omega$ if and only if ω is integral.*

Therefore, the prequantizability of a symplectic manifold is equivalent to the integrality of its symplectic structure. Cohomologically the integrality of ω is described by the following diagramm :

$$\begin{array}{ccccccc} \mathfrak{L}(M) & \xrightarrow{c} & \check{H}^2(M, \mathbb{Z}) & \xrightarrow{\varepsilon} & \check{H}^2(M, \mathbb{R}) & \xleftarrow{d.R.} & H^2(M, \mathbb{R}) \\ [L] & \longrightarrow & c[L] & \longrightarrow & \varepsilon(c[L]) = d.R.[\omega] & \longleftarrow & [\omega] \end{array} \tag{3}$$

where $\mathfrak{L}(M)$ is the Picard's group of M, i.e. the family of all equivalence classes of complex line bundles over M endowed with a certain group structure, $\check{H}^2(M,\mathbb{Z})$ is the second $\check{C}$ech cohomology group of M with values in $\mathbb{Z}$, $\check{H}^2(M,\mathbb{R})$ is the second $\check{C}$ech cohomology group of M with values in $\mathbb{R}$, and $H^2(M,\mathbb{R})$ is the second de Rham cohomology group of M with values in $\mathbb{R}$.

For a prequantizable symplectic manifold (M,ω) the algebra $(C^\infty(M,\mathbb{R}),\{\cdot,\cdot\}_\omega)$ is then represented in the Hilbert space $\mathcal{H} = L^2(L,\omega^n,<\cdot,\cdot>)$ of all square ω^n-integrable smooth sections of L via

$$C^\infty(M,\mathbb{R}) \ni f \longmapsto \mathcal{Q}_{KS}(f) = -i \bigtriangledown_{X_f} + f \in Sym(\mathcal{H}). \tag{4}$$

Clearly, the map $\mathcal{Q}_{KS}$ is prequantization in the sense of Def. 2.1. Nevertheles, it is in general not a quantization. With the help of Stone's theorem one can show that $\mathcal{Q}_{KS}(f)$ is self-adjoint whenever X_f is complete.

Now we proceed to look for the local expression for the operator $\mathcal{Q}_{KS}(f)$ for any classical observable f. We start from

Proposition 2.1 *Let $(L,\bigtriangledown)$ be a complex line bundle with connection over a symplectic manifold (M,ω) and $\{(U_\alpha,s_\alpha)\}$, where α's are in an index set, is a local system of L with transition functions $\{c_{\alpha\beta}\}$. Then on every $U_\alpha \subset M$ there exists a smooth complex 1-form θ_α so that for every vector field X on U_α*

$$\bigtriangledown_X s_\alpha = -i(X \lrcorner \theta_\alpha)s_\alpha. \tag{5}$$

Furthermore we have the gauge transformation

$$\theta_\beta = \theta_\alpha - \frac{dc_{\alpha\beta}}{ic_{\alpha\beta}}. \tag{6}$$

Conversely, if a set of 1-forms $\{\theta'_\alpha\}$ satisfies the gauge transformation Eq.(6) then there exists a unique connection $\bigtriangledown'$ so that $\bigtriangledown'_X s_\alpha = -i(X \lrcorner \theta'_\alpha)s_\alpha$ for every vector field X on U_α.

Proposition 2.2 *Let L^* the pricipal line bundle associated to $(L,\bigtriangledown)$. Then there exists a unique 1-form θ on L^*, i.e. the connection form of $\bigtriangledown$, so that $\theta_\alpha = s^*_\alpha\theta$ for every $(U_\alpha,s_\alpha) \in \{(U_\alpha,s_\alpha)\}$.*

Proposition 2.3 *$(L,\bigtriangledown)$ admits a hermitian structure so that $\bigtriangledown$ is a hermitian connection if and only if the real 1-form $i(\theta - \bar{\theta})$ on L^* is exact.*

Remark 2.1 *For a prequantization line bundle $(L,\bigtriangledown,<\cdot,\cdot>)$ every local 1-form θ_α on U_α which satisfies Eq.(5) is a local symplectic potential, $d\theta_\alpha = \omega \restriction U_\alpha$.*

Hence, for an observable $f \in C^\infty(M,\mathbb{R})$ we have locally

$$\mathcal{Q}_{KS}(f)s \restriction U_\alpha = -iX_f(\psi)s_\alpha - (X_f \lrcorner \theta_\alpha)\psi s_\alpha + f\psi s_\alpha \tag{7}$$

for every $s \in \mathcal{H}$, where ψ is a complex function on U_α so that $s \restriction U_\alpha = \psi s_\alpha$.

In the case of $Q = \mathbb{R}^n$, one already encounters some essential problems in obtaining the Schrödinger representation : **(i)** The operators $\mathcal{Q}_{KS}(p_i)$ and $\mathcal{Q}_{KS}(q^i)$ are not the usual momentum and position operators of quantum mechanics, **(ii)** The kinetic energy function $T = (1/2)\Sigma_i p_i^2$ is maped under $\mathcal{Q}_{KS}$ into $\mathcal{Q}_{KS}(T)\psi = -i\sum_i p_i(\partial\psi/\partial q^i) - T\psi$, for every $\psi \in L^2(\mathbb{R}^{2n}, d^npd^nq)$, and **(iii)** The representation Hilbert space is to 'large'. It contains wave functions which depend on both position and momentum.

Then, one has to reduce the size of the representation Hilbert space $\mathcal{H}$. The reduction of $\mathcal{H}$ is controled from 'below' by reducing the classical phase space (M, ω) through *polarization*, namely a foliation of M whose leaves are *Lagrangian submanifolds* with respect to ω. The rigorous definition is

Definition 2.2 *Let $T^{\mathbb{C}}M$ denote the complexified tangent bundle of M and $\mathfrak{X}^{\mathbb{C}}(M)$ denote the set of all smooth complex vector fields on M. A polarization P on (M, ω) is a subbundle of $T^{\mathbb{C}}M$ which respects the following requirements :*

$P1$ *Any $X, Y \in P_m$ satisfy $\omega_m(X, Y) = 0$ for every $m \in M$, where P_m means the fibre of P over m,*

$P2$ *The space $\mathfrak{X}_P(M) := \{X \in \mathfrak{X}^{\mathbb{C}}(M) : X(m) \in P_m,\ \forall m \in M\}$ is closed under the Lie bracket of vector fields,*

$P3$ *The distribution $m \mapsto D^{\mathbb{C}}(m) := P_m \cap \bar{P}_m$ is of constant dimension for every $m \in M$.*

From the polarization P one can derive another distribution beside $D^{\mathbb{C}}$, namely $m \mapsto E^{\mathbb{C}}(m) := P_m + \bar{P}_m$. It follows that the set $\mathfrak{X}_D$ of all smooth sections of D is closed under Lie bracket of vector fields. The closedness of $\mathfrak{X}_D$ is equivalent to the integrability of D. Let M/D denote the space of all leaves of D and π_D the canonical projection $\pi_D : M \longrightarrow M/D$.

Definition 2.3 *A polarization P is called reducible whenever the following conditions are satisfied : (a) E is a foliation, (b) M/D and M/E are smooth manifolds, and (c) The projection $\pi_E : M/D \longrightarrow M/E$ is a submersion.*

The smooth manifold M/D is called the *generalized configuration space extracted from* (M, ω) by P. We call a polarization P *real* whenever $P = \bar{P}$ and Kähler whenever $P \cap \bar{P} = 0$.

With the polarization at hand one constructs the so-called $-\frac{1}{2}$-P-densities line bundle $(|\wedge^n|^{-\frac{1}{2}}B^P(M), \tilde{\bigtriangledown})$ with partial flat connection. Furthermore, the quantum bundle $\mathcal{B}_P$ is defined as the tensor product $(L \otimes |\wedge^n|^{-\frac{1}{2}}B^P(M), \bigtriangledown \otimes \tilde{\bigtriangledown})$ between the prequantization line bundle $(L, \bigtriangledown)$ and the bundle $(|\wedge^n|^{-\frac{1}{2}}B^P(M), \tilde{\bigtriangledown})$. Then the Hilbert space $\mathcal{H}_P$ of representation is constructed from the polarized sections of $\mathcal{B}_P$. i.e. $\mathcal{H}_P$ is the completion of

$$\mathcal{W}_P := \{\psi \in \mathcal{S}(\mathcal{B}_P) : (\bigtriangledown_P \otimes \tilde{\bigtriangledown}_P)\psi = 0 \ \text{and} \ \int_{M/D} (\psi, \psi)_P < \infty\} \tag{8}$$

endowed with scalar product

$$<\psi, \psi'>_P := \int_{M/D} (\psi, \psi')_P, \quad \forall \psi, \psi' \in \mathcal{H}_P, \tag{9}$$

where $(\cdot,\cdot)_P$ is a 1-density-valued pairing in the set of all polarized sections of $\mathcal{B}_P$. An observable $f \in C^\infty(M,\mathbb{R})$ is called *quantizable* whenever $[X_f, Y] \in \mathfrak{X}_P(M)$ for every $Y \in \mathfrak{X}_P(M)$, where $\mathfrak{X}_P(M)$ is the set of all vector fields on M with $X(m) \in P_m$, $\forall m \in M$. The observable f is quantized via

$$\mathcal{Q}_P(f)\psi = -i(\nabla_{X_f} s) \otimes \nu + f(s \otimes \nu) - is \otimes \mathfrak{L}_{X_f}\nu, \quad \forall \psi = s \otimes \nu \in \mathcal{H}_P, \tag{10}$$

where $\mathfrak{L}$ is Lie derivative in $\mathcal{S}(|\wedge^n|^{-\frac{1}{2}} B^P(M))$. Applying this procedure to $(\mathbb{R}^{2n}, \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i})$ with *vertical* polarization yields the usual momentum and position operators of quantum mechanics.

Let $\mathcal{F}_P(M,\mathbb{R})$ denote the set of all P-quantizable observables. Then we have in general $\mathcal{F}_P(M,\mathbb{R}) \subset C^\infty(M,\mathbb{R})$. Even in the case of the cotangent bundle T^*Q of a smooth orientable n-dimensional manifold Q with the vertical polarization P_v the quantizable subalgebra $\mathcal{F}_{P_v}(T^*Q,\mathbb{R})$ contains of functions of the form $\zeta \circ pr + \Sigma_i^n(\zeta^i \circ pr)p_i$, where ζ, ζ^i $(i = 1, \cdots, n)$ are functions contained in $C^\infty(Q,\mathbb{R})$ and $(p_1, \cdots, p_n, q^1 \cdots, q^n)$ is a (local) canonical coordinat system on T^*Q. Thus, the kinetic energy is not contained in the quantizable subalgebra. Therefore, the theory is in general still lacking of dynamic. The problem is solved partially by using the so-called BKS-transform ([18, 14, 13]). Applying this formalism to the cotangent bundle $\pi : T^*Q \to Q$ of an orientable geodatically complete manifold (Q, g) one gets

$$i\partial_t\psi = -\frac{1}{2}\Delta_g\psi + (V + \frac{1}{12}R)\psi \tag{11}$$

for energy function $H = (1/2) < g^\sharp(\alpha), \alpha > + \pi^*V(\alpha)$, $\alpha \in T^*Q$, with certain potential $V \in C^\infty(Q,\mathbb{R})$, where Δ_g the Laplace-Beltrami operator associated to g and R the scalar curvature of g.

3 Geometric Quantization on Poisson Manifolds

Let (M, Λ) a Poisson manifold. Now define a mapping $l_1 : \Omega^1(M) \to \mathfrak{X}(M)$ from the space of 1-forms on M into the space of smooth vector fields on M as follows :

$$l_1(\alpha)(\beta) = \Lambda(\alpha, \beta) \tag{12}$$

for $\alpha, \beta \in \Omega^1(M)$. It is clear that if M is symplectic, then l_1 is an isomorphism. The mapping l_1 can be extended to a mapping $l_p : \Omega^p(M) \to \mathfrak{X}_p(M)$ from the space of p-forms on M into the space of p-vector on M via

$$l_0(f) = f \qquad l_p(\alpha)(\alpha_1, \cdots, \alpha_p) = (-1)^p\alpha(l_1(\alpha_1), \cdots, l_1(\alpha_p)) \tag{13}$$

for $f \in C^\infty(M,\mathbb{R})$, $\alpha \in \Omega^p(M)$, and $\alpha_1, \cdots, \alpha_p \in \Omega^1(M)$. For every function $f \in C^\infty(M,\mathbb{R})$ then there exists a smooth vector field $X_f = l_1(df)$. X_f is called the ***Hamiltonian vector field*** or ***Hamiltonian equation of motion*** generated by f. The mapping $\delta : \mathfrak{X}_p(M) \to \mathfrak{X}_{p+1}(M)$ defined by $\mathfrak{X}_p(M) \ni \Upsilon \longmapsto \delta\Upsilon = -[\Upsilon, \Lambda] \in \mathfrak{X}_{p+1}(M)$ is contravariant exterior derivative. Since $\delta^2 = 0$, δ defines a cohomology on M, the ***Lichnerowicz-Poisson (LP) cohomology*** for the Poisson manifold M (see [10]). Since Λ is a δ-cocycle, therefore Λ defines a LP cohomology class $[\Lambda] \in H^2_{LP}(M)$. Furthermore, the mapping l_p induced homomorphism $l_p : H^p_{dH}(M) \to H^p_{LP}(M)$.

Definition 3.1 *An orientable*[3] *Poisson manifold* (M, Λ) *is called* *prequantizable* *if there exists a complex line bundle* $(L, \bigtriangledown, < \cdot, \cdot >)$ *over* M *with hermitian connection* $\bigtriangledown$ *so that the curvature* $\Omega \equiv Cur(L, \bigtriangledown)$ *satisfies* $l_2(\Omega) = \Lambda$.

The prequantizability of (M, Λ) is equivalent to the condition that the LP class $[\Lambda]$ is *integral*, i.e. that the space $d.R.(l_2^{-1}[\Lambda])$ is contained in the image $\varepsilon(\check{H}^2(M, \mathbb{Z}))$ of $\check{H}^2(M, \mathbb{Z})$ under the natural map $\varepsilon : \check{H}^2(M, \mathbb{Z}) \longrightarrow \check{H}^2(M, \mathbb{R})$ induced by the injection $j : \mathbb{Z} \hookrightarrow \mathbb{R}$, where $d.R. : H^2(M, \mathbb{R}) \longrightarrow \check{H}^2(M, \mathbb{R})$ is the de Rham isomorphism.

The orientability of M then implies the existence of a smooth volume form on M. Let $\mathcal{S}(L)$ denote the set of all smooth sections of L and μ be a volume form on M. Then the completion $\mathcal{H}$ of

$$\mathcal{S}_0(L) := \{s \in \mathcal{S}(L) : \int_M < s, s > \mu < \infty\} \tag{14}$$

endowed with the scalar product

$$(s, s') := \int_M < s, s' > \mu, \quad \forall s, s' \in \mathcal{H} \tag{15}$$

is a Hilbert space. It is easy to see that the mapping $\mathcal{Q}_{KSP} : C^\infty(M, \mathbb{R}) \to Sym(\mathcal{H})$ given by

$$\mathcal{Q}_{KSP}(f) = -i \bigtriangledown_{X_f} + f \tag{16}$$

is a prequantization.

For a complex pfaffian system $\mathcal{E}$ on M let $\mathcal{E}^\perp$ denote the pfaffian system on M obtained from $\mathcal{E}$ by

$$\mathcal{E}^\perp_m = \{\alpha \in T^*_m M^{\mathbb{C}} : \Lambda_m(\alpha, \beta) = 0, \quad \forall \beta \in \mathcal{E}_m\} \tag{17}$$

for all $m \in M$. Then, we propose the following

Definition 3.2 *A* *complex polarization* $\mathcal{E}$ *on a Poisson manifold* (M, Λ) *is a smooth complex pfaffian system on* M *of maximal rank so that*

P1 *for every* $m \in M$ *and* $\alpha, \beta \in \mathcal{E}_m$ *gilt* $\Lambda_m(\alpha, \beta) = 0$,

P2 *the distribution* $P \equiv l_1(\mathcal{E}^\perp)$ *given by* $P_m = \{\Lambda_m(\alpha, \cdot) : \alpha \in \mathcal{E}^\perp_m\}$ *is integrable, and*

P3 $D^{\mathbb{C}}_m \equiv P_m \cap \overline{P}_m$ *is of constant dimension for all* $m \in M$.

The polarization $\mathcal{E}$ *is called* *reducible* *whenever the set* M/D *of all maximal integral manifolds of* D *is a smooth manifold. The manifold* M/D *is called* *generalized configuration space extracted by* $\mathcal{E}$ *from* M.

By making use of the mapping l_1 it is easy to show that Def.3.2 reduces to the old definition of polarization on symplectic manifold, whenever M is symplectic.

[3] In the case of symplectic manifolds, the orientability is clearly already guaranted.

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