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AN EXAMPLE OF SYSTEM WHICH CAN BE USED TO EXPLICITLY SHOW THE SYMMETRY BETWEEN THE ELECTRIC AND MAGNETIC FIELDS

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Abstract

Electric and magnetic fields are almost but not exactly symmetric. That fact does not preclude the existence of perfect symmetry in specific systems in a certain context. In this paper we describe an example of such systems which are charge distributions with cylindrical symmetry and their related systems of current distributions with cylindrical symmetry (which of course can be realized by moving the first systems with constant velocity longitudinally along the cylindrical length). We will show that in these systems there is perfect symmetry and we will prove that the symmetry can be understood from the standpoint of special relativity.

Key words: symmetry, electric and magnetic fields, special relativity

INTRODUCTION

One of the disturbing problems students must face when studying electromagnetism is that they have to deal with two vector fields (electric and magnetic fields) with different properties. In fact the two fields follow different equations so that they must be different in general situations. We can somehow decrease the burden if we can show that the two fields are symmetric or analogous (in some intuitive sense) in some special situations.

In this paper we intend to do just that : showing that the electric and magnetic fields are the analogues of each other in certain systems. Then we will show that special relativity can explain the reason behind the symmetry. Finally we give an example of the application of this symmetry and presenting a somewhat unusual derivation of the field of sheet of charge using the electric potential.

THE SYMMETRY BETWEEN ELECTRIC AND MAGNETIC FIELDS

The most fundamental equations governing electricity and magnetism are Maxwell's equations (Panofsky and Phillips, 1962).

$$div\,\vec{E} = \frac{\rho}{\epsilon_0} \qquad (1)$$

$$rot\,\vec{E} = 0 \qquad (2)$$

$$div\,\vec{B} = 0 \qquad (3)$$

$$rot\,\vec{B} = \mu_0 \vec{J} \qquad (4)$$

We can see that in general, equations 1 and 2 (electricity) are different from equations 3 and 4 (magnetism). For certain special situations, in places where there are no electric charges and currents then the right hand sides of equations 1 and 4 will be zero and equation 1 is becoming identical with equation 3, and also equation 2 with 4. In those situations then there is some hope that we can find systems with symmetries (in the intuitive sense) between electric and magnetic fields.

Let us look at an infinite wire of charge which can be taken to be the z axis. From basic physics we have derived the electric field to be pointing radially (cylindrical) outward.

$$\vec{E} = \frac{1}{r} = \frac{\vec{r}}{r^2} \quad (5)$$

We suppressed the constants to make the discussion simpler. If we look at the system from the positive direction of the z axis, then the wire will look like a point and we deal with a two dimensional system.

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} \quad (6)$$
$$E_x = x/r^2 \quad (7)$$
$$E_y = y/r^2 \quad (8)$$

We can verify that the field is indeed irrotational.

$$\frac{\partial}{\partial x}\left(E_y\right) = \frac{\partial}{\partial y}\left(E_x\right) = -2\,\frac{x\,y}{(x^2+y^2)^2} \quad (9)$$

We can find the potential by using partial integration.

$$\Phi_e = -\int E_x\,\partial x = -\int E_y\,\partial y = \int E_r\,dr = -ln(r) \quad (10)$$

The contour lines of the potential are concentric circles with the center at the origin of coordinates. The electric field can be determined from the potential using

$$\vec{E} = -grad\;\Phi_e \quad (11)$$

which in polar system of coordinate can be written as

$$\vec{E} = -\left(\frac{d\Phi_e}{dr}\right)\hat{r} \quad (12)$$

We look at a wire (which can be taken as the z axis) carrying a current. From basic physics we have already derived the magnetic field as

$$\vec{B} = \hat{\theta}/r \quad (13)$$

pointing tangentially. Using Cartesian coordinate system we get the components of the magnetic field as

$$B_x = -y/r^2 \quad (14)$$

$$B_y = x/r^2 \tag{15}$$

We can prove that the magnetic field is irrotational.

$$\frac{\partial}{\partial x}\left(B_y\right) = \frac{\partial}{\partial y}\left(B_x\right) = \frac{y^2 - x^2}{(x^2+y^2)^2} \tag{16}$$

We can determine the magnetic potential

$$\Phi_m = -\int B_x\, \partial x = -\int B_y\, \partial y = -\int B\ r d\theta = -\theta = -arctan\left(\frac{y}{x}\right) \tag{17}$$

In this case the magnetic field can be determined in identical manner as the electric field.

$$\vec{B} = -grad\ \Phi_m \tag{18}$$

In this point of view, both fields follow the same law regarding the relation between the field and the potential. The difference is in the potentials with contour lines intersecting perpendicularly (concentric circles of the electric and radial lines of the magnetic). This is one sense of symmetry. There is a weakness in the interpretation. The existence of potential means that the integral of

$$\int_A^B B_t\, ds \tag{19}$$

between two points does not depend on the path connecting the two points (Zeldovich, 1976). The value of the integral is equal to the difference between the values of the potentials at the two points. Whereas for the electric case the integral has the physical meaning as the work done by the field in moving one unit of electric charge, in the magnetic case it does not have any clear or significant physical meaning.

We can use another line of attack by using the magnetic vector potential which in this special case has only the z component that can be regarded as the magnetic scalar potential. We can verify that the divergence of the magnetic field is zero.

$$\frac{\partial}{\partial x}\left(B_x\right) = -\frac{\partial}{\partial y}\left(B_y\right) = \frac{2xy}{(x^2+y^2)^2} \tag{20}$$

We can determine the magnetic scalar potensial as the z component of the vector potential.

$$\Phi_m = A_z = \int B_x\, \partial y = -\int B_y\, \partial x = -ln(r) \tag{21}$$

The magnetic field can be determined from the scalar potential using

$$\vec{B} = B_x\,\hat{\imath} + B_y\,\hat{\jmath} = \frac{\partial \Phi_m}{\partial y}\,\hat{\imath} - \frac{\partial \Phi_m}{\partial x}\,\hat{\jmath} \tag{22}$$

which can be converted into polar coordinate system

$$\vec{B} = -\left(\frac{d\Phi_m}{dr}\right)\hat{\theta} \qquad (23)$$

In this approach the value of the scalar potentials (the electric and the magnetic) are the same. The difference is in the law of the relation between the field and the potential (in which the form of the derivatives are the same, but the direction of the unit vectors are perpendicular). That is another sense of the symmetry. The existence of the magnetic scalar potential has a clear physical meaning.

$$\int_A^B B_n\, ds \qquad (24)$$

The integral between two points does not depend on the path and the integral is the magnetic flux crossing the path. So the value of the magnetic flux crossing any path connecting two points is equal to the difference of the magnetic scalar potentials at the two points.

According to special relativity, if there is electric field in the reference frame K then there must be magnetic field in the reference frame K1 moving at constant velocity with respect to K.

$$\vec{B} = \vec{V}\, X\, \vec{E} \qquad (25)$$

with ignoring constants to simplify the discussion.

In our case the frame K is home for the infinite wire of charge. Frame K1 is moving with constant velocity in the positive z direction. In K1 the charge wire becomes the infinite wire carrying current. Distances perpendicular to the z axis are not affected by the transformation of frames so that from the point of view of the positive z axis, the sole effect of the transformation is the electric field giving rise to magnetic field perpendicular to the electric field in accord to equation 25. This is exactly what equation 23 says. The magnetic field can be determined from the electric field just by changing the unit vector by another unit vector perpendicular to the former.

AN EXAMPLE OF APPLICATION

An infinite sheet of charge can be represented by $x = 0$ and from the point of view of the positive z axis, the sheet is represented as a line (the y axis). The electric field must be pointing along the x axis due to symmetry so that the field can be determined by integrating all contributions from the line charge along the y axis.

$$E_x = \int_{-\infty}^{\infty}\left(\frac{x}{x^2+y^2}\right)dy = \pi * sign(x) \qquad (26)$$

where the value of *sign(x)* is +1 or -1 in accord with the positive or negative value of x. We get the result that the electric field is a constant vector pointing along the x axis.

For a current sheet flowing to the positive z axis we can conclude that the magnetic field is a constant vector pointing along the y axis. The factor of the sign of the axis in equation 26 is interpreted as the fact that there is a discontinuity of the field when we cross the sheet.

Using the scalar potential turns out to be more complicated. This is presumably the reason that this way has not been explored in textbooks. The resulting integral is not simple such that we need the assistance of Computer Algebra System (CAS). One of the most convenient to solve calculus problems is the CAS Derive (which is unfortunately has been discontinued although there are still so many enthusiasts until now) (Magiera, 2001). One of the main features of this CAS is that it is primarily to be used as sophisticated symbolic calculator. This feature makes it very convenient as an important tool to assist the mathematical derivations in theoretical physics.

We have to use a trick (which is often used by physicist in tackling mathematical problems (Feyerabend, 1992)). The trick is to place the field point on the x axis close enough to the source (which is the y axis). We assume the wire to be of finite unit length on both direction of the y axis. In the original problem the field point is at finite distance from infinite wire. We transform the situation into an equivalent one in which we have a finite wire with the field point at infinitely close distance to the wire.

$$\Phi = \int_{-1}^{1} ln\left(\frac{1}{(x^2+y^2)^{\frac{1}{2}}}\right) dy = -2x\ arctan\left(\frac{1}{x}\right) - ln(x^2+1) + 2 \qquad (27)$$

We then take the Taylor series in *x*.

$$\Phi = -\pi * sign(x) * x - \frac{x^4}{6} + x^2 + 2 \qquad (28)$$

We have to ignore all smaller terms when *x* approaching zero. We take two terms because the first term is a constant which means that there is no field but from the physics of the problem there has to be some field. The next smaller term is linear in *x* and we ignore all other smaller terms. Finally we get the result as

$$\Phi = -\pi * sign(x) * x \qquad (29)$$

exactly similar to equation (26) with the consideration of equation 18.

CONCLUSION AND SUGGESTION

We have shown that in certain special systems the electric and magnetic fields have nice symmetry (in intuitive sense). This fact can somehow give more insight to the relation between electric and magnetic fields, and can be employed to derive certain important results.

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