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GRAM-SCHMIDT SUPER ORTHOGONALIZATION PROCESS FOR SUPER LINEAR ALGEBRA

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Abstract

Gram-Schmidt Process is a method to transform an arbitrary basis into an orthogonal basis then normalize the orthogonal basis vectors to obtain an orthonormal basis. This process is so important and has many uses in applications of mathematics, particularly linear algebra and numerical analysis. Super linear algebra is an extension of linear algebra, in the which talks about the super matrices, super vectors up to super basis, super orthogonal basis and super diagonalization on a super inner product super spaces. It will be discussed a process to construct an arbitrary basis into an super orthogonal and orthonormal basis for super inner-product super spaces. The modification of the Gram-Schmidt Process to construct an super orthogonal and orthonormal basis, namely Gram-Schmidt Orthogonalization Process for Super Super Linear Algebra.

Key words: Gram-Schmidt Super Orthogonalization Process and Super Linear Algebra.

INTRODUCTION

Gram-Schmidt Process is a method to transform an arbitrary basis into an orthogonal basis then normalize the orthogonal basis vectors to obtain an orthonormal basis [1]. This process is so important and has many uses in applications of mathematics, particularly linear algebra and numerical analysis [9].

The study of super matrices started in the year 1963 by Paul Horst. His book on matrix algebra speaks about super matrices of different types and their applications to social problems [3]. These new structures can be applied to all fields in which linear algebras are used. These new structures can be applied to Computer Science, Markov chains, and Fuzzy Models [4].

Super linear algebra is an extension of linear algebra, in the which talks about the super matrices, super vectors up to super basis, super orthogonal basis and super diagonalization on a super inner product super spaces. The supermatrix and supervector are one whose elements are themselves matrices or vector with elements that can be either scalars or other matrices or vector (see [3], [4], [5]). The supermatrix and supervector can be constructed from a simple matrix or simple vector, this process of constructing supermatrix is called the partitioning (see [3]). The definition of super vector spaces can be seen in [4].

SUPER LINEAR ALGEBRA

Every super linear algebra is a super vector space but in general every super vector space need not be a super linear algebra. The following is the definition of super linear algebra;

DEFINITION 1: [4]

Let $V = (V_1| \ldots |V_n)$ be a super vector space over a field F. We say V is a super linear algebra over F if and only if for every pair of super row vectors α, β in V the product of α and β denoted by $\alpha\beta$ is defined in V in such a way that

(a) multiplication of super vector in V is associative i.e., if α, β and $\gamma \in V$ then $\alpha(\beta\gamma) = (\alpha\beta)\gamma$.

(b) multiplication is distributive $(\alpha + \beta)\gamma = (\alpha\gamma + \beta\gamma)$ and $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ for every $\alpha, \beta, \gamma \in V$.

(c) for each scalar c in F $c(\alpha\beta) = (c\alpha)\beta = \alpha(c\beta)$.

If there is an element 1_e in V such that $1_e\alpha = \alpha 1_e$ for every $\alpha \in V$ we call the super linear algebra V to be a super linear algebra with identity over F. The super linear algebra V is called commutative if $\alpha\beta = \beta\alpha$ for all α and β in V.

The definition of linear combination of super vector, super subspace, super subspace spanned, linearly independent of super vector and super basis can be seen in [4] is similar to usual vector spaces in [7] and [1] with only change in case of super vector spaces they occur in n-tuples.

SUPER BASIS

DEFINITION 2: [4]

Let V be a super vector space over the field F. A super basis or simply a basis for V is clearly a dependent set of super vectors V which spans the space V. The super space V is finite dimensional if it has a finite basis.

LEMMA 1: [4]

Let S be a linearly independent subset of a super vector space V. Suppose β is a vector in V and not in the super subspace spanned by S, then the set obtained by adjoining β to S is linearly independent.

THEOREM 1: [4]

If W is a super subspace of a finite dimensional super vector space V, every linearly independent subset of W is finite and is part of a (finite basis for W).

COROLLARY 1: [4]

If W is a proper super subspace of a finite dimensional super vector space V, then W is finite dimensional and $\dim W < \dim V$.

COROLLARY 2: [4]

In a finite dimensional super vector space V every non empty linearly independent set of super vectors is part of a basis.

SUPER INNER PRODUCT SUPER SPACES

The definition of super inner product super spaces which we call as super inner product spaces in [4].

DEFINITION 3: [4]

Suppose that $V = (F^{n_1}| \ldots |F^{n_1})$ be a super inner product space over the field F. Then for $\alpha \in$ V with $\alpha = \left(\alpha_1^1 \ldots \alpha_{n_1}^1\middle| \ldots \middle|\alpha_1^n \ldots \alpha_{n_n}^n\right)$ and $\beta \in V$ where $\beta = \left(\beta_1^1 \ldots \beta_{n_1}^1\middle| \ldots \middle|\beta_1^n \ldots \beta_{n_n}^n\right)$

$$(\alpha|\beta) = \left(\sum_{j1} \alpha_{j1}\bar{\beta}_{j1} | \ldots | \sum_{jn} \alpha_{jn}\bar{\beta}_{jn}\right).$$

DEFINITION 4: [4]

The super norm of $\alpha = (\alpha_1| \ldots |\alpha_n) \in V = (V_1| \ldots |V_n)$ define by

$$\sqrt{(\alpha|\alpha)} = \left(\sqrt{(\alpha_1|\alpha_1)}\middle| \ldots \middle|\sqrt{(\alpha_n|\alpha_n)}\right) = (\|\alpha_1\|| \ldots |\|\alpha_n\|) = \|\alpha\|.$$

The super quadratic form determined by the inner product is the function that assigns to each super vector α the scalar n-tuple $\|\alpha\|^2 = (\|\alpha_1\|^2 | \dots | \|\alpha_n\|^2)$. Hence just like an inner product space the super inner product space is a real or complex super vector space together with a super inner product on that space.

SUPER ORTHOGONAL SUPER SET AND SUPER ORTHONORMAL SUPER SET

The definition of super orthogonal super set and super orthonormal super set is similar to usual vector spaces in [7] and [1] with only change in case of super vector spaces they occur in n-tuples.

DEFINITION 5: [4]

Let $\alpha = (\alpha_1 | \dots | \alpha_n)$ and $\beta = (\beta_1 | \dots | \beta_n)$ be super vectors in a super inner product space $V = (V_1 | \dots | V_n)$. Then α is **super orthogonal** to β if $(\alpha|\beta) = \big((\alpha_1|\beta_1), \dots, (\alpha_n|\beta_n)\big) = (0| \dots |0)$ since this implies β is super orthogonal to α, we often simply say α and β are super orthogonal. If $S = (S_1 | \dots | S_n)$ is a super subset of super vectors in $V = (V_1 | \dots | V_n)$ S is called a **super orthogonal super set** provided all pairs of distinct super vectors in S are super orthogonal i.e. by the super orthogonal subset we mean every set S_i in S is an orthogonal set for every $i = 1, 2, \dots, n$ i.e. $(\alpha_i|\beta_i) = 0$ for all $\alpha_i, \beta_i \in S$; $i = 1, 2, \dots, n$. A **super orthonormal super set** is a super orthogonal set with additional property $\|\alpha\| = (\|\alpha_1\| | \dots | \|\alpha_n\|) = (1| \dots |1)$, for every α in S and every α_i in S_i is such that $\|\alpha_i\| = 1$.

From the definition we get the following theorem and corollary;

THEOREM 2: [4]

A super orthogonal super set of nonzero super vector is linearly super independent.

COROLLARY 3: [4]

If a super vector $\beta = (\beta_1 | \dots | \beta_n)$ is a linear super combination of orthogonal sequence of non-zero super vectors, $\alpha_1, \alpha_2, \dots, \alpha_m$ then β in particular is a super linear combination,

$$\beta = \left(\sum_{K_1=1}^{m_1} \frac{(\beta_1|\alpha_{K_1}^1)}{\|\alpha_{K_1}^1\|^2} \alpha_{K_1}^1 \,\middle|\, \dots \,\middle|\, \sum_{K_n=1}^{m_n} \frac{(\beta_n|\alpha_{K_n}^n)}{\|\alpha_{K_n}^n\|^2} \alpha_{K_n}^n \right).$$

DEFINITION 6: [4]

Let $V = (V_1 | \dots | V_n)$ be an inner product super space and S any set of super vectors in V. The super orthogonal complement of S is the superset $S^\perp$ of all super vectors in V which are super orthogonal to every super vector in S.

DEFINITION 7: [4]

Let $V = (V_1 | \dots | V_n)$ be an inner product super space over the field F. Let $W = (W_1 | \dots | W_n)$ be a super subspace of a super inner product super space V and let $\beta = (\beta_1 | \dots | \beta_n)$ be a super vector in V. $\alpha = (\alpha_1 | \dots | \alpha_n)$ in W is called the orthogonal super projection to $\beta = (\beta_1 | \dots | \beta_n)$ on $W = (W_1 | \dots | W_n)$. If every super vector in V has an orthogonal super projection of $\beta = (\beta_1 | \dots | \beta_n)$ on W, the mapping that assigns to each super vector in V its orthogonal super projection on $W = (W_1 | \dots | W_n)$ is called the orthogonal super projection of V on W. Suppose $E_S = (E_1 | \dots | E_n)$ is the orthogonal super projection of V on W. Then the super mapping $\beta \to \beta - E_S\beta$ i.e., $\beta = (\beta_1 | \dots | \beta_n) \to (\beta_1 - E_1\beta_1 | \dots | \beta_n - E_n\beta_n)$ is the orthogonal super projection of V on $W^\perp = (W_1^\perp | \dots | W_n^\perp)$.

THEOREM 3: [4]

Let $V = (V_1 | \dots | V_n)$ be a super inner product space and let $\left(\beta_1^1 \middle| \dots \middle| \beta_{n_1}^1\right), \dots, \left(\beta_1^n \middle| \dots \middle| \beta_{n_n}^n\right)$ be any independent super vector in V. Then one may construct orthogonal super vector $\left(\alpha_1^1 \middle| \dots \middle| \alpha_{n_1}^1\right), \dots, \left(\alpha_1^n \middle| \dots \middle| \alpha_{n_n}^n\right)$ in V such that for each $K = (K_1 | \dots | K_n)$ the set

$\{(\alpha_1^1|\ldots|\alpha_{K_1}^1),\ldots,(\alpha_1^n|\ldots|\alpha_{K_n}^n)\}$ is a super basis for the super subspace spanned by $(\beta_1^1|\ldots|\beta_{K_1}^1),\ldots,(\beta_1^n|\ldots|\beta_{K_n}^n)$.

THEOREM 4: [4]

Every finite $(n_1,n_2,\ldots,n_n)$ dimensional super inner product superspace has an orthonormal super basis.

DISCUSSION

Our next theorem, which is the main result shows that every nonzero finite-dimensional super vector space has an super orthonormal basis. The proof of this result is extremely important, since it provides an algorithm, or method, for converting an arbitrary super basis into an super orthogonal basis or super orthonormal basis. To proof the following theorem will proceed by the Principle of Mathematical Induction/recursion [2] and will use a modified proof of Paul Skoufranis [8].

THEOREM (The Gram-Schmidt Super Orthogonalization Process for Super Linear Algebra)

Let V be an super inner product super space and let $S=(\alpha_1|\ldots|\alpha_n)$ be a super subset of V. Define $S'=(\beta_1|\ldots|\beta_n)$ recursively by $\beta_1=\alpha_1$ and

$$\beta_k=\alpha_k-\sum_{j=1}^{k-1}\frac{(\alpha_k|\beta_j)}{\|\beta_j\|^2}\beta_j$$

for all $2\le k\le n$ $\left(where\ \frac{(\alpha_k|\beta_j)}{\|\beta_j\|^2}\beta_j=\vec{0}_V\ \ if\ \beta_j=\vec{0}_V\right)$. Then S' is an super orthogonal subset of V such that $super\ span(S)=super\ span(S')$. Moreover, if S is super linearly independent, then S' is super linearly independent $\left(and\ thus\ \beta_j\neq 0\ for\ all\ j=1,\ldots,n\right)$ and

$$\mathfrak{B}=\left(\frac{\beta_1}{\|\beta_1\|}\middle|\frac{\beta_2}{\|\beta_2\|}\middle|\ldots\middle|\frac{\beta_n}{\|\beta_n\|}\right)$$

is an super orthonormal basis for $super\ span(S)$.

Proof: To begin the proof of the first claim, we will proceed by the Principle of Mathematical Induction/recursion (we do not need the full power of the Principle of Mathematical Induction since the number of steps will be finite) on the mathematical statements P_m that $(\beta_1|\ldots|\beta_m)$ is an super orthogonal subset such that $super\ span((\beta_1|\ldots|\beta_m))=super\ span((\alpha_1|\ldots|\alpha_m))$.

Base Case: $m=1$ By definition $\beta_1=\alpha_1$ so the result is trivially true.

Inductive/Recursive Step Suppose that the result is true for some m;

that is, $(\beta_1|\ldots|\beta_m)$ is an super orthogonal subset such that $super\ span((\beta_1|\ldots|\beta_m))=super\ span((\alpha_1|\ldots|\alpha_m))$. We desire to prove the result for $m+1$ provided that $m+1\le n$ (as if $m=n,$ we are done). Thus suppose further that $m+1\le n$.

To see that $(\beta_1|\ldots|\beta_{m+1})$ is an super orthogonal subset, we need to show that $(\beta_i|\beta_j)=0$ whenever $i\neq j$. Since $(\beta_1|\ldots|\beta_m)$ is an super orthogonal subset, we need to only check that $(\beta_{m+1}|\beta_i)=0$ for each $i=1,\ldots,m$. However, if $i\in\{1,\ldots,m\}$, then

$$\begin{aligned}(\beta_{m+1}|\beta_i) &= \left(\alpha_{m+1}-\sum_{j=1}^{m}\frac{(\alpha_{m+1}|\beta_j)}{\|\beta_j\|^2}\beta_j\middle|\beta_i\right)\\ &= (\alpha_{m+1}|\alpha_i)-\sum_{j=1}^{m}\frac{(\alpha_{m+1}|\beta_j)}{\|\beta_j\|^2}(\beta_j|\beta_i)\end{aligned}$$

$= (\alpha_{m+1}|\alpha_i) - \frac{(\alpha_{m+1}|\beta_j)}{\|\beta_j\|^2}\|\beta_i\|^2$; $(\beta_j|\beta_i) = 0$ unless $j = i$ by the induction hypothesis

$= (\alpha_{m+1}|\alpha_i) - (\alpha_{m+1}|\beta_i) = 0$

as desired. Hence $(\beta_1| \dots |\beta_{m+1})$ is an super orthogonal subset of V.

To see that $super\ span((\beta_1| \dots |\beta_{m+1})) = super\ span((\alpha_1| \dots |\alpha_{m+1}))$, we note that $super\ span((\beta_1| \dots |\beta_m)) = super\ span((\alpha_1| \dots |\alpha_m))$ by assumption. Therefore, to prove the desired equality, it is enough to show that $\beta_{m+1} \in super\ span((\alpha_1| \dots |\alpha_{m+1}))$ and $\alpha_{m+1} \in super\ span((\beta_1| \dots |\beta_{m+1}))$. To see that $\beta_{m+1} \in super\ span((\alpha_1| \dots |\alpha_{m+1}))$, we recall that

$$\beta_{m+1} = \alpha_{m+1} - \sum_{j=1}^{m} \frac{(\alpha_{m+1}|\beta_j)}{\|\beta_j\|^2}\beta_j$$

However,since
$super\ span((\beta_1| \dots |\beta_m)) = super\ span((\alpha_1| \dots |\alpha_m)), \beta_j \in super\ span((\alpha_1| \dots |\alpha_m))$ for all $j = 1, \dots, m$ and thus

$$\beta_{m+1} = \alpha_{m+1} - \sum_{j=1}^{m} \frac{(\alpha_{m+1}|\beta_j)}{\|\beta_j\|^2}\beta_j \in super\ span((\alpha_1| \dots |\alpha_{m+1})).$$

To see that $\alpha_{m+1} \in super\ span((\beta_1| \dots |\beta_{m+1}))$, we notice, by rearranging the above equation, that

$$\alpha_{m+1} = \beta_{m+1} + \sum_{j=1}^{m} \frac{(\alpha_{m+1}|\beta_j)}{\|\beta_j\|^2}\beta_j \in super\ span((\beta_1| \dots |\beta_{m+1})).$$

which completes the inductive/recursive step.

Hence, by the Principle of Mathematical Induction (or recursion), we have proven the first part of the theorem.

Next suppose that S is a super linearly independent set. Let $W = super\ span(S)$. Therefore W is a finite dimensional super subspace of . Since S is a super linearly independent super subset with n super vectors and $W = super\ span(S)$, S is a super basis for W and thus super $\dim(W) = n$. However, since $super\ span(S') = super\ span(S) = W$ and S' has exactly n super vectors, S' must be a super linearly independent super subset of W by a corollary to the Replacement Theorem. Therefore $\beta_j \neq \vec{0}_V$ for all $j = 1, \dots, n$ as S' is super linearly independent and thus does not contain the zero super vector. Therefore $\|\beta_j\| \neq 0$ for all j so $\mathfrak{B}$ is a well-defined set of super vectors.

Finally, to see that $\mathfrak{B}$ is an super orthonormal basis for W, we notice that, since S' is an super orthogonal set,

$$\langle \frac{\beta_i}{\|\beta_i\|}, \frac{\beta_j}{\|\beta_j\|} \rangle = \frac{1}{\|\beta_i\|\|\beta_j\|}(\beta_i|\beta_j) = 0 \text{ for all } i \neq j \text{ and}$$

$$\left\| \frac{\beta_j}{\|\beta_j\|} \right\| = \frac{1}{\|\beta_j\|}\|\beta_j\| = 1 \text{ for all } j = 1, \dots, n$$

so that $\mathfrak{B}$ is an super orthonormal set of super vectors. Since $\mathfrak{B}$ contains n super vectors and super $\dim(W) = n$, we see that $\mathfrak{B}$ is an super orthonormal basis for W as desired. □

In the preceding proof we used the Gram–Schmidt process to produce an super orthogonal basis; then, after the entire super orthogonal basis was obtained, we normalized to obtain an super orthonormal basis. Alternatively, one can normalize each super orthogonal basis vector as soon as it is obtained, there by generating the super orthonormal basis step by step such modifications are made in [6]. However, this method has the slight disadvantage of producing more square roots to manipulate.

The modification of the Gram-Schmidt Process to construct an super orthogonal and orthonormal basis above, namely **Gram-Schmidt Orthogonalization Process for Super Linear Algebra**. We get;
Let V be an super inner product super space and let $S = (\alpha_1| \dots |\alpha_n)$ be a super subset of V. Define $S' = (\beta_1| \dots |\beta_n)$
Step 1 : $\beta_1 = \alpha_1$
Step 2 : $\beta_k = \alpha_k - \sum_{j=1}^{k-1} \frac{(\alpha_k|\beta_j)}{\|\beta_j\|^2} \beta_j$ *for all* $2 \leq k \leq n$ *and*
Step 3: $\mathfrak{B} = \left(\frac{\beta_1}{\|\beta_1\|} \middle| \frac{\beta_2}{\|\beta_2\|} \middle| \dots \middle| \frac{\beta_n}{\|\beta_n\|}\right)$ *is an super orthonormal basis.*

CONCLUSION AND SUGGESTION

Super linear algebra is an extension of linear algebra, in the which talks about the super matrices, super vectors up to super basis, super orthogonal basis and super diagonalization on a super inner product super spaces. The modification of the Gram-Schmidt Process to an arbitrary basis into an super orthogonal and orthonormal basis for super inner-product super spaces, namely Gram-Schmidt Orthogonalization Process for Super Super Linear Algebra. We get;
Let V be an super inner product super space and let $S = (\alpha_1| \dots |\alpha_n)$ be a super subset of V. Define $S' = (\beta_1| \dots |\beta_n)$
Step 1 : $\beta_1 = \alpha_1$
Step 2 : $\beta_k = \alpha_k - \sum_{j=1}^{k-1} \frac{(\alpha_k|\beta_j)}{\|\beta_j\|^2} \beta_j$ *for all* $2 \leq k \leq n$ *and*

Step 3: $\mathfrak{B} = \left(\frac{\beta_1}{\|\beta_1\|} \middle| \frac{\beta_2}{\|\beta_2\|} \middle| \dots \middle| \frac{\beta_n}{\|\beta_n\|}\right)$ *is an super orthonormal basis.*

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