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OPTIMAL CONTROL ANALYZE AND EQUILIBRIUM EXISTENCE OF SEIR EPIDEMIC MODEL WITH BILINEAR INCIDENCE AND TIME DELAY IN STATE AND CONTROL VARIABLES

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Abstract

In this paper, we will formulate S E I R epidemic model with bilinear incidence and time delay as basic model. Then, we will investigate the optimal control strategy of our model with time delay in state and control variables. We use a vaccination program as optimal control strategy to minimize the number of susceptible and to maximize the recovered individuals. By mathematical analysis we find the existence of solution and optimal control of our model. We use Pontryagin's maximum principle to characterize this optimal control.

Keywords : SEIR, Time Delay, Optimal Control, Vaccination

1. INTRODUCTION

S E I R epidemiological models divide total host population into four classes that is susceptible , exposed, infectious and removed classes. For various reasons, in epidemiological models, the real dynamic behaviour of an epidemic depends not only on its current state but also on its past history (Elhia, et. al.[3]). Thus, many epidemiological model including delay time to model the different aspect of epidemics, like latent period. In epidemic problem, one of the strategies to control the dynamics of respective disease is vaccination strategies. To understand and analyze this problem, we use optimal control approach to minimize the number of susceptible, exposed and invected individual and to maximize the number of recovered individuals during the course of an epidemic. In this paper, we will investigate the optimal control problem with delay in simple S E I R model with bilinear incidence. We adding time delay as parameter into state and control variable. In order to optimal vaccination control problem, we use objective functional with quadratic term control. Our goal is to prove the existence of optimal control for simple S E I R model with billinear incidence βSI (where $\beta > 0$ is the average number of contacts per infective per unit time) and to find those with characterize properties by Pontryagin's maximum (or minimum) principle. Futher more we hope that all the result in this paper can be made as referency to furher research of optimal control vaccination in generalized SEIR model with more complex assumption.

2. MATHEMATICAL MODEL

In this paper, we use SEIR epidemic model with constant of recruitments and bilinear incidendce We can find The detail about background and assumption about this model in [1] and [2]. We assume there is a different natural death rate in each compartment. There is also death rate, which is denoted by σ, that is induced by disease, but. We can get a SEIR epidemiological model as follows

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t)I(t) - \mu_1 S(t)\\
\frac{dE(t)}{dt} &= \beta S(t)I(t) - (\varepsilon + \mu_2)E(t)\\
\frac{dI(t)}{dt} &= \varepsilon E(t) - (\gamma + \mu_3 + \sigma)I(t)\\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu_4 R(t)\\
N(t) &= S(t) + E(t) + I(t) + R(t)
\end{aligned} \tag{2.1}$$

with initial conditions $S(0) = S_0, E(0) = E_0, I(0) = I_0, R(0) = R_0$ and the total host population is $N(t) = S(t) + E(t) + I(t) + R(t)$. Optimal control techniques are great of use in deveoping optimal strategies to control various kind of disease. One of these strategies is vaccination strategies. Our goal is to reduce the numbers of susceptible, exposed and invected, and then increase the number of recovered individuals.

For construct the control optimal problem, we need to define the suitable control variable u, that is indicate the percentage of susceptible individuals being vaccinated per unit of time. Accoding to the result by Elhia, et.al [3], We will define new parameter, that is account the movement of the vaccinated individuals from the class of susceptibles into the recovered class is subject to delay. Then, the time delay τ introduced in the system as follows : at time t only a per-centage of susceptible individuals that have been vaccinated τ time unit ago, that is, at time $t-\tau$, are removed from the susceptible class and added to the recovered class. Hence, we get a new S E I R model with time delay in control and state variable as follows :

$$\begin{aligned}
\frac{dS(t)}{dt} &= \Lambda - \beta S(t)I(t) - \mu_1 S(t) - u(t-\tau)s(t-\tau)\\
\frac{dE(t)}{dt} &= \beta S(t)I(t) - (\varepsilon + \mu_2)E(t)\\
\frac{dI(t)}{dt} &= \varepsilon E(t) - (\gamma + \mu_3 + \sigma)I(t)\\
\frac{dR(t)}{dt} &= \gamma I(t) - \mu_4 R(t) + u(t-\tau)s(t-\tau)\\
N(t) &= S(t) + E(t) + I(t) + R(t)
\end{aligned} \tag{2.2}$$

Based on biological assumption, we assume that $\theta \in [-\tau, 0]$, it is implies that $S(\theta), E(\theta), I(\theta), R(\theta)$ is non negative real valued function and $u(\theta) = 0$. In this model, we also assume that u is lebesque measurable and $0 \le u \le b < 1$, where b is determined coeffisient

3. THE EXISTENCE OF SOLUTIONS

Before we prove the existence of solutions of system (2.2), we need to prove that system (2.2) is a dissipative system, on the otherhand, we can say all the feasible solutions of system (2.2) is uniformly bounded in Ω that is subset of $\mathbb{R}^4_+$. Let $(S, E, I, R) \in \mathbb{R}^+$ is any solutions of system (2..2) with non negative initial values. If we adding firt, second and third equation of equation (2.2), we get

$$\frac{dN}{dt} = \Lambda - \mu_1 S(t) - \mu_2 E(t) - (\mu_3 + \sigma)I(t) - \mu_3 R(t) < \Lambda - \mu N(t) \qquad (3.1)$$

with $\mu = min(\mu_1, (\mu_2 + \sigma), \mu_3)$. After integration, using the constant variation formula , we have

$$N(t) \leq \frac{\Lambda}{\mu} + N(0)e^{-\mu t} \qquad (3.2)$$

that is implies

$$0 \leq N(t) \leq \frac{\Lambda}{\mu}, \quad t \to \infty \qquad (3.3)$$

Hence, all the feasible solutions of sytem (2.2) is already exist and uniformly bounded in region Ω

$$\Omega = \left\{(S, E, I, R) \in \mathbb{R}_+^4 : N \leq \frac{\Lambda}{\mu}\right\} \qquad (3.4)$$

Then, we can rewrite system (2.2) in the following form in new variable X

$$\frac{dX}{dt} = AX + F(X, X_\tau) = G(X, X_\tau) \qquad (3.5)$$

where

$$X(t) = \begin{bmatrix} S(t) \\ E(t) \\ I(t) \\ R(t) \end{bmatrix}$$

$$A = \begin{bmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & -(\varepsilon + \mu_2) & 0 & 0 \\ 0 & \varepsilon & -(\gamma + \mu_3 + \sigma) & 0 \\ 0 & 0 & \gamma & -\mu_4 \end{bmatrix}$$

$$F\big(X(t), X_\tau(t)\big) = \begin{bmatrix} \Lambda - \beta S(t)I(t) - u_\tau(t)s_\tau(t) \\ \beta S(t)I(t) \\ 0 \\ u_\tau(t)s_\tau(t) \end{bmatrix}$$

$$X_\tau(t) = X_\tau(t - \tau), \;\; u_\tau = u(t - \tau)$$

We have

$$\left|F\big(X_1(t), X_{1\tau}(t)\big) - F\big(X_2(t), X_{2\tau}(t)\big)\right| \leq M_1|X_1(t) - X_2(t)| + M_2|X_{1\tau} - X_{2\tau}| \quad (3.6)$$

where M_1 and M_2 are positive real constant that is independent from state variables variables $S(t), E(t), I(t)$dan $R(t)$, and we have

$$|X_1(t) - X_2(t)| = |S_1(t) - S_2(t)| + |E_1(t) - E_2(t)| + |I_1(t) - I_2(t)| + |R_1(t) - R_2(t)|$$

$$|X_{1\tau}(t) - X_{2\tau}(t)| = |S_{1\tau}(t) - S_{2\tau}(t)| + |E_{1\tau}(t) - E_{2\tau}(t)| + |I_{1\tau}(t) - I_{2\tau}(t)| + |R_{1\tau}(t) - R_{2\tau}(t)|$$

where

$$S_{i\tau}(t) = S_i(t-\tau), E_{i\tau}(t) = E_i(t-\tau), I_{i\tau}(t) = I_i(t-\tau), \quad R_{i\tau}(t) = R_i(t-\tau),$$

for $i = 1,2$, we can get

$$|G(X_1, X_{1\tau}) - G(X_2, X_{2\tau})| \leq M(|X_1 - X_2| + |X_{1\tau} - X_{2\tau}|) \quad (3.7)$$

with $M = max(M_1 + \|A\|, M_2)$. Based on equation (3.7), we can conclude that G is uniformly Lipschitz continous. From the definiton of $u(t)$ and the value of batasan nilai $S(t), E(t), R(t), I(t)$, then the solutions of system (2.2) is exist.

4. THE EXISTENCE OF EQUILIBRIUM AND THEIR LOCAL STABILITY

In this chapter, we will find the equilibrium of the system (2.2) that is disease free and disease equilibrium.

Definition 4.1. (Packard, Poola, Horowitz, [5])

Consider a system of nonlinear differentian equation

$$\dot{x}(t) = f\big(x(t), u(t)\big) \quad (4.1)$$

Where f is function mapping $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. A point $\bar{x} \in \mathbb{R}^n$. is called an equilibrium point of system (4.1) if there is a specific $\bar{u} \in \mathbb{R}^m$ (called the equilibrium input) such that

$$f(\bar{x}, \bar{u}) = 0_n$$

Suppose $\bar{x}$ is an equilibrium point (with equilibrium input $\bar{u}$). Consider starting the system (4.1) from initial condition $x(t_0) = \bar{x}$, and applying the input $u(t) \equiv \bar{u}$ for all $t \geq t_0$. The resulting solution x(t) satisfies

$$x(t) = \bar{x}$$

for all $t \geq t_0$

Consider the above definition, we will derive the existence of the equilibrium of system (2.2). The condition without control is $u(t-\tau) = u^* = 0$. Without loss of generality, we take $\tau = 0$ to fulfill those condition.Then, we get the equilibrium of system (2.2) is same with the equilibrium that is can be obtained from following equation

$$\Lambda - \beta S(t)I(t) - \mu_1 S(t) = 0 \quad (1)$$
$$\beta S(t)I(t) - (\varepsilon + \mu_2)E(t) = 0 \quad (2)$$
$$\varepsilon E(t) - (\gamma + \mu_3 + \sigma)I(t) = 0 \quad (3)$$
$$\gamma I(t) - \mu_4 R(t) = 0 \quad (4)$$

(.2)

We get $\boldsymbol{E_0} = \left(\frac{\boldsymbol{\Lambda}}{\boldsymbol{\mu_1}}, \boldsymbol{0}, \boldsymbol{0}, \boldsymbol{0}\right)$ and it is always exist. Then, from equation (4) and (5) we get

$$R^* = \frac{\gamma}{\mu_4} I^* \text{ and } E^* = \frac{(\gamma + \mu_3 + \sigma)}{\varepsilon} I^*$$

Then, from the first and second equation we get

$$\Lambda - \mu_1 S(t) = (\varepsilon + \mu_2)E(t)$$

$$\Leftrightarrow \Lambda - \mu_1 S^* = \frac{(\gamma + \mu_3 + \sigma)}{\varepsilon} I^*$$

For $I^* = 0$, we get the disease free equilibrium $\boldsymbol{E_0} = \left(\frac{\boldsymbol{\Lambda}}{\boldsymbol{\mu_1}}, \mathbf{0}, \mathbf{0}, \mathbf{0}\right)$**.** If $I^* \neq 0$ then

$$\frac{\varepsilon\Lambda - \varepsilon\mu_1 S^*}{(\gamma + \mu_3 + \sigma)} = I^*$$

$$I^* = 1 - \frac{1}{R_0}$$

It's implies

$$S^* = \frac{\Lambda}{\mu_1} - \frac{(\gamma + \mu_3 + \sigma)}{\varepsilon\mu_1}\left(1 - \frac{1}{R_0}\right)$$

For $R_o > 1$ we get disease equilibrium $E^* = (S^*, E^*, I^*, R^*)$, where

$$S^* = \frac{\Lambda}{\mu_1} - \frac{(\gamma + \mu_3 + \sigma)}{\varepsilon\mu_1}\left(1 - \frac{1}{R_0}\right)$$

$$E^* = \frac{(\gamma + \mu_3 + \sigma)}{\varepsilon}\left(1 - \frac{1}{R_0}\right)$$

$$I^* = 1 - \frac{1}{R_0}$$

$$R^* = \frac{\gamma}{\mu_4}\left(1 - \frac{1}{R_0}\right)$$

5. THE OPTIMAL CONTROL PROBLEM AND THE EXISTENCE OF OPTIMAL CONTROL

5.1. MODEL FORMULATION

Our goal in this research is to minimize the number of susceptible, exposed and also invected individuals and to maxime the number of recovered individuals with vaccination strategy. Obviously, for fixed terminal time t_{end}, the problem is minimize the objective functional with quadratic term control

$$J = \int_0^{t_{end}} \left\{ w_1 S(t) + w_2 E(t) + w_3 I(t) - w_4 R(t) + \frac{1}{2} w_5 u^2(t) \right\} dt \tag{5.1}$$

where $w_i, i = 1,2,3,4,5$ denoted weights that balance the size of the terms.Then, we will find optimal control $u^* \in \mathcal{U}$ such that

$$J(u^*) = min(J(u) \colon u \in \mathcal{U}) \tag{5.2}$$

where $\mathcal{U}$ is the set of admissible controls defined by

$$\mathcal{U} = \{u \colon 0 \leq u \leq u_{max} < \infty, t \in [0, t_{end}], u \text{ Lebesgue Measurable}\} \tag{5.3}$$

5.2. THE EXISTENCE OF OPTIMAL CONTROL

In this section, we will find the optimal control of system (2.2) by using the result of a result by Fleming and Rishel in Ehia, et. al , [3]

Theorem 5.1

Consider the control problem with system (2.2) there is an optimal `control $u^* \in \mathcal{U}$ such that

$$J(u^*) = min(J(u): u \in \mathcal{U}) \tag{5.4}$$

Proff.

Based on the result in chapter 3 about the existence of the solutions, we get the information that the sets of control and the corresponding state variables is nonempty. By the definition of control set (5.3), then the control set $\mathcal{U}$ is convex and closed. Since The state system is linear in u, the right side of system (2.2) is bounded by a linear function in the state and control variables. It's clear that the integrand in the objective functional (5.3) is convex on $\mathcal{U}$. In addition, there is exist a constant $\rho > 1$ and also positive numbers ω_1 and ω_2 satisfiying

$$L(S, E, I, R, u) \geq \omega_1 + \omega_2(|u|^2)^{1/\rho}$$

where L is Lagrangian in the objective functional (3.5). Based on the above properties, we can conclude that there is an optimal `control $u^* \in \mathcal{U}$ satisfiying equation (5.4)

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