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SOME PROPERTIES OF PRIMITIVE ϑ −HENSTOCK OF INTEGRABLE FUNCTION IN LOCALLY COMPACT METRIC SPACE OF VECTOR VALUED FUNCTION

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Abstract
By abstracting the domain and range of Henstock integrable function and measures (the volume function) used by mathematicians, we had constructed ϑ −Henstock integral in locally compact metric space of vector valued function. Based on the result of previous researchers, this paper discuss properties of primitve ϑ −Henstock integrable function in locally compact metric space of vector valued function.

Key words: ϑ −Henstock integrable function , locally compact metric space, vector.

1. INTRODUCTION

1.1 Background

In 1989, Lee had contructed Henstock integral of real valued fungtion on a cell $[a, b] \subset \mathbb{R}$. Manuharawati (2002) generalized this integral by reduced a cell $[a, b] \subset \mathbb{R}$ to a cell in a locally compact metric space and Cao (1992) generalized this integral by reduced the range of function from $\mathbb{R}$ to Banach space.

Based on the result of these researchers, Manuharawati and Yunianti (2013) had contructed a Henstock integral of vector valued function (the range of function is a Banach space) in locally compact metric space. In this integral, we used a volume function ϑ that defined on the set of all elementary sets in a locally compact metric space. In this paper, we discuss some properties of the primitive this integral.

1.2 Fundamental Concepts

Let X be a locally compact metric space, $\mathbb{R}^*$ be an extended real numbers, $\mathbb{R}^+$ be a positive real numbers, and $E(X)$ be a collection of elementary sets in X.

A nonvoid collection $S \subset 2^X$ is called a **system of intervals** if it satisfies: (Manuharawati, 2002: 11-12):

(i) for every $p \in X$, $\{p\} \in S$,

(ii) for every $p \in X$, if $cl(N(p,r))$ compact then $N(p,r) \in S$,

(iii) if $A \in S$ then A konvex, $cl(A)$ compact, $cl(A) \in S$, and $int(A) \in S$,

(iv) for every $A, B \in S$, $A \cap B \in S$.

(v) for every $A, B \in S$ there exists a collection of nonoverlapping sets $C_i \in S$, $1 \leq i \leq n$ such that

$$A - B = \bigcup_{i=1}^{n} C_i.$$

Every an element of S is called an **interval**. Interval $A \in S$ is said to be **degenerate** if $int(A) = \emptyset$ and **nondegenerate** if $int(A) \neq \emptyset$. A **cell** is a nondegenerate compact interval. A set $A \subset X$ is called **elementary set** if A is a finite union of intervals.

A Volume function $\vartheta: E(X) \to \mathbb{R}$ is called a measure on $E(X)$ if:

(i) $\vartheta(A) = 0$ if $int(A) = \emptyset$ and $\vartheta(A) > 0$ if $int(A) \neq \emptyset$

(ii) $\vartheta(A) \leq \vartheta(B)$ for every $A, B \in E(X)$ with $A \subset B$,

(iii) $\vartheta(\bigcup_{i\in\mathbb{N}} A_i) = \sum_{i\in\mathbb{N}} \vartheta(A_i)$ for every nonoverlapping collection sets $\{A_i\} \subset E(X)$ with $\bigcup_{i\in\mathbb{N}} A_i \in E(X)$.

In this paper X is a nondiscrete locally compact metric space.

If E is a cell and $\delta: E \to \mathbb{R}^+$, then a finite collection of cell-point pairs

$$P = \{(A_{x_i}, x_i), 1 \leq i \leq n\} = \{(A_x, x)\}$$

is called a **Perron δ-*fine* partition on** E if $x_i \in A_{x_i} \subset N(x_i, \delta(x_i))$ and $\{A_{x_i}\}$ is a partition on E.

If $E \subset X$ is a cell and $\delta: E \to \mathbb{R}^+$, then there exists a Perron δ-*fine* partition on E (Manuharawati, 2002: 18).

A real function $\|.\|$ on a vector space Y (over field K) is called a norm on Y if for every $x, y \in Y$ and $\alpha \in K$ satisfies:

(i) $\|x\| \geq 0$,
(ii) $\|x\| = 0 \Longrightarrow x = \boldsymbol{O}$,
(iii) $\|\alpha X\| = |\alpha| \, \|x\|$,
(iv) $\|x + y\| \leq \|x\| + \|y\|$.

A vector space Y thegether a norm is called a norm space. A Banach space is a complete norm space.

A function $g: X \to Y$ is said to be Henstock ϑ- integrable on a cell $E \subset X$ there exists a vector $a \in Y$ such that for every real number $\varepsilon > 0$ thre exists a function $\delta: E \to \mathbb{R}^+$ such that for every Perron δ-fine partition

$$P = \{(A_{x_i}, x_i), 1 \leq i \leq n\} = \{(A_x, x)\}$$

on E we have

$$\left\|\sum_{i=1}^{n} g(x_i)\, \vartheta(A_{x_i}) - a\right\| = \|P \sum g(x)\; \vartheta(A_x) - a\| < \varepsilon$$

If $A \in S$ with $\vartheta(A) = 0$, then integral- ϑ Henstock of a function $g: X \to Y$ on A is defined by null vectorl ($\boldsymbol{O}$). The set of all ϑ −Henstock integrable function on a cell E is denoted by

$H(E,\vartheta)$ and a vector a is called ϑ −Henstock integral value of a function f on E and denoted by

$$(H-\vartheta)\int f = a.$$

(Manuharawati and Yunianti, 2013: 20-21).

Some properties of the ϑ −Henstock integrable function which will be used in discussion are as follow.

Theorem 1.2.1: Cauchy's Criterian (Manuharawati and Yunianti, 2013a: 71-75): *A functin $g \in H(E,\vartheta)$ if and only if for any real number $\varepsilon > 0$ there exists a function $\delta : E \to \mathbb{R}^+$ such that for any Perron δ-fine partition*

$$P = \{(A_x, x)\} \text{ and } Q = \{(B_y, y)\}$$

on E we have

$$\left\| P\sum g(x)\ \vartheta(A_x) - Q\sum g(y)\ \vartheta(B_y) \right\| < \varepsilon.$$

Theorem 1.2.2: (Manuharawati and Yunianti, 2013: 24) *If $g \in H(E,\vartheta)$, Then for every a cell $F \subset E$, $g \in H(F,\vartheta)$.*

Theorem 1.2.3: (Manuharawati, 2013: 22-26) *A function $g: X \to Y$ is Henstock ϑ- integrable* on a *cell $E \subset X$ if only if there exists an additive function $G: I(E) \to Y$ with property for any real number $\varepsilon > 0$ there exists a function $\delta : E \to \mathbb{R}^+$ such that for any Perron δ-fine partition*

$$P = \{(A_{x_i}, x_i), 1 \le i \le n\} = \{(A_x, x)\}$$

on E,

$$\left\| \sum_{i=1}^{n} g(x_i)\,\vartheta(A_{x_i}) - G(E) \right\| = \| P\sum g(x)\ \vartheta(A_x) - G(E) \| < \varepsilon.$$

If E is a cell and $I(E)$ is a collection of all subinterval in E, then an additive function $G: I(E) \to Y$ is called ϑ −Henstock primitive of f on E if $G(\emptyset) = \mathbf{0}$ (null vektor) and for every $A \in I(E)$, $A \neq \emptyset$, we have

$$G(A) = (H-\vartheta)\int_{Cl(A)} g.$$

Theorem 1.2.4: Henstock's Lemma (Manuharawati, 2013: 22-26): *If $g: X \to Y$ $\vartheta -$ Henstock integrable on a cell $E \subset X$ with primitive G, then for any real number $\varepsilon > 0$ there exists a function $\delta: E \to \mathbb{R}^+$ shuch that for any Perron δ-fine partition $P = \{(A_x, x)\}$ and any $Q \subset P$ we have*

$$\|Q \sum [g(x)\vartheta(A_x) - G(A_x)]\| < 2\varepsilon.$$

2. RESULT AND DISCUSSION

Let X be a nondicret metric space, Y be Banach space, ϑ be a volume function on $E(X)$, and E be a cell in X. We start define an interval continuous function relative to volume function ϑ.

Definition 2.1: *A function $G: I(E) \to Y$ is said to be $\upsilon -$continuous in an interval $A \in I(E)$ if for every real number $\varepsilon > 0$ there exist a real number $\gamma_A > 0$ such that for every $B, C \in I(E)$ with $B \subset A \subset C$, $\upsilon(A - B) < \gamma_A, \upsilon(C - A) < \gamma_A$ we have*

$$\|G(A - B)\| < \varepsilon \text{ and } \|G(C - A)\| < \varepsilon.$$

A function G is said to be $\upsilon -$continuous on $(I(E)$ if G is $\vartheta -$continuous in every interval $A \in I(E)$.

Lemma 2.2: *If an additive function $G: I(E) \to Y$ is $\upsilon -$continuous on $I(E)$ then for every real number $\varepsilon > 0$ there exist a real number $\gamma > 0$ such that for every interval $A \in I(E)$ with $\upsilon(A) < \gamma$ we have*

$$\|G(A)\| < \varepsilon.$$

Proof: Let $\varepsilon > 0$ be given. Since G is $\upsilon -$continuous on $I(E)$, then G is $\upsilon -$continuous at $\{x\}$ for every $x \in E$. So for every $x \in E$ there exist a real number $\gamma_x > 0$ such that for every open interval $O_x \in I(E)$ with $x \in O_x$ and $\vartheta(O_x) < \gamma_x$ we

$$\|G(O_x)\| < \varepsilon.$$

If

$$\mathcal{L}_1 = \{O_x \in I(E): x \in int(E), x \in O_x, O_x \text{ is open, and } \vartheta(O_x) < \gamma_x\}$$

$$\mathcal{L}_2 = \{B_x \in S: x \in \partial(E),\ x \in B_x, B_x \text{ is open, and } \vartheta(B_x \cap E) < \gamma_x\},$$

then

$$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2.$$

is a vitali cover of E. Since E is compact set, then we have $x_1, x_2, \dots, x_n \in E$ such that

$$\{O_{x_i} \in \mathcal{L}_1: 1 \le i \le l\} \cup \{O_{x_i} \in \mathcal{L}_1: 1 \le i \le l\}\{B_{x_i} \in \mathcal{L}_2: 1 \le i \le m\}$$

with $l + m = n$ is an open cover of E.

Further, put

$$\gamma = min\{\gamma_{x_i}: 1 \le i \le n\}.$$

Since G is an additive function, then for every interval $A \in I(E)$ with $\upsilon(A) < \gamma$ we have

$$\|G(A)\| < \varepsilon. \quad \blacksquare$$

Definition 2.3: *A function $G: I(E) \to Y$ is said to be strongly absolutely υ −continuous on $F \subset E$ if for every real number $\varepsilon > 0$ there exist a real number $\gamma > 0$ such that for every partition $\{A_1, A_2, A_3, \dots, A_n\}$ in E with $\partial(A_i) \cap F \ne \emptyset$* and $\sum_{i=1}^{n} \vartheta(A_i) < \gamma$ we have

$$\sum_{i=1}^{n} \|G(A_i)\| < \varepsilon.$$

Theorem 2.4: *Let a function $g: X \to Y$ be given. If $g \in H(E, \vartheta)$ with ϑ −Henstock primitve of g on E is G and g is bounded, then G is trongly absolutly υ −continuous on E.*

Proof: From boundednes of g on E, there exist a real number $M > 0$ such that for every $x \in E$ implies

$$\|g(x)\| \le M.$$

Given any real number $\varepsilon > 0$. Based on density property on $\mathbb{R}$, there exists a real number $\eta > 0$ such that $\eta < \frac{\varepsilon}{2M}$. Take a collection $\{A_i, i = 1, 2, \dots, n\}$, $A_i \subset E$, with

$$\sum_{i=1}^{n} \upsilon(A_i) < \eta.$$

Since $g \in H(E, \vartheta)$, then by Henstock's Lemma and Theorem 1.2.3, there exist a function $\delta: E \to \mathbb{R}^+$ such that for every δ −fine partition $P_i\{(A_x, x)\}$ on A_i we have

$$\left\| P_i \sum [G(A_x - g(x)\upsilon(A_x)] \right\| < \frac{\varepsilon}{2^{i+1}}.$$

Since G is a υ −Henstock primitive of g on E, then by Theorem 1.2.2 for every i we have

$$\begin{aligned} \|G(A_i)\| &= \|P_i \textstyle\sum G(A_x)\| \\ &\le \|P_i \textstyle\sum G(A_x) - g(x)\upsilon(A_x)\| + \|P_i \textstyle\sum g(x)\upsilon(A_x)\| \\ &< \tfrac{\varepsilon}{2^{i+1}} + M\,\upsilon(A_i). \end{aligned}$$

So we have

$$\begin{aligned} \sum_{i=1}^{n} \|G(A_i)\| &< \sum_{i=1}^{n} \left[\tfrac{\varepsilon}{2^{i+1}} + M\,\upsilon(A_i)\right] \\ &< \sum_{i=1}^{n} \tfrac{\varepsilon}{2^{i+1}} + M \sum_{i=1}^{n} \upsilon(A_i) \\ &< \tfrac{\varepsilon}{2} + M\,\tfrac{\varepsilon}{M} = \varepsilon. \end{aligned}$$

This mean that G is strongly absolutly ϑ −continuous on E. $\blacksquare$

Theorem 2.5: *Let a function $g: X \to Y$be given. If $g \in H(E, \vartheta)$ with ϑ −Henstock primitve of g on E is G, then $G$is ϑ −continuous on $I(E)$.*

Proof: Let ε be areal positive number. If A is a cell in $I(E)$, then there exists open set O_A in X such that $A \subset O_A$ and $\vartheta(O_A - A) < \varepsilon$. If $\mathcal{L}$ be a collection of all O_A with $A \subset O_A$ for all $A \in I(E)$, then $\mathcal{L}$ is an open cover of E. Since E is a compact set, then there exists

$$\mathcal{M} = \{O_{A(i)}, i = 1, 2, \dots, n\} \subset \mathcal{L}$$

such that

$$E \subset \bigcup_{i=1}^{n} O_{A(i)}$$

If $F \in I(E)$, then by Lemma 2.2 and additivity of G, there exists areal number $\gamma_F > 0$ such that if $\vartheta(F) < \gamma_F$ we have

$$\|G(F)\| < \frac{\varepsilon}{n}.$$

Let

$$\gamma = \min\{\gamma_A : O_A \in \mathcal{M}\}$$

and $B, C, D \in I(E)$ be cells with

$$B \subset C \subset D, \vartheta(C - B) < \gamma \text{ and } \vartheta(D - C) < \gamma.$$

Then there exist two finite collection of nonoverlapping cells

$$\{C_i : i = 1, 2, \ldots, l\} \subset I(E) \text{ and } \{D_i : i = 1, 2, \ldots, s\} \subset I(E)$$

such that

$$cl(C - B) = \bigcup_{i=1}^{l} C_i \text{ and } cl(D - C) = \bigcup_{i=1}^{s} D_i.$$

Since $\vartheta(C - B) < \gamma$ and $\vartheta(D - C) < \gamma$, then for eny i, $\vartheta(C_i) < \gamma$ and $\vartheta(D_i) < \gamma$. Note that, G is an additive function on $I(E)$. So, we have

$$\|G(C - B)\| = \sum_{i=1}^{l} \|G(C_i)\| < l\frac{\varepsilon}{n} \text{ and } \|G(D - C)\| = \sum_{i=1}^{l} \|G(D_i)\| < s\frac{\varepsilon}{n}.$$

So, G is ϑ −continuous on $I(E)$. ■

Theorem 2.6: *Let a function $g: X \to Y$be given. If $g \in H(E, \vartheta)$ with ϑ −Henstock primitve of g on E is G, then $D_\upsilon G(x)$ exists allmost everywhere on E and $D_\upsilon G(x) = g(x)$ allmost everywhere on E.*

Proof: Given a real number $\varepsilon > 0$. Since $g \in H(E, \vartheta)$ with ϑ −Henstock primitive is G, then by Henstock's Lemma, there exist a function $\delta: E \to \mathbb{R}^+$ such that for every Perron δ −fine partition $P = \{(A_x, x)\}$ on E and every subpartition P_0 of P we have

$$P_0 \sum \|G(A_x) - g(x)\upsilon(A_x)\| < \varepsilon.$$

If

$$Z = \{x \in E : D_\upsilon G(x) \text{ exist or } D_\upsilon G(x) \text{ exist but } D_\upsilon G(x) \neq g(x)\},$$

then for every $x \in Z$ there exists a real number $\eta(x) > 0$ with property: for every real number $\gamma > 0$ there exists $B_x \in E(x, \gamma, \eta(x))$ shuch that

$$\|G(B_x) - g(x)\upsilon(A_x)\| > \eta(x)\upsilon(B_x).$$

For every $k \in \mathbb{N}$, put

$$Z_k = \left\{x \in Z : \eta(x) > \frac{1}{k}\right\}.$$

So wehave

$$Z = \bigcup_{k \in \mathbb{N}} Z_k.$$

If

$$V = \left\{B_x : x \in Z_k,\ B_x \in E\big(x, \gamma, \eta(x)\big), \gamma < \delta(x)\right\},$$

then V is is vitali cover for Y_k. So, there exists a nonoverlapping collection of cell

$$\left\{B_{x_i}; i = 1,2, \ldots, p\right\} \subset V$$

such that

$$\begin{aligned} \upsilon^*(Y_k) &\leq \sum_{i=1}^{p} \upsilon\big(B_{x_i}\big) + \varepsilon \\ &< k \sum_{i=1}^{p} \big\|G\big(B_{x_i}\big) - g(x)\upsilon(B_{x_i})\big\| + \varepsilon \\ &< k\varepsilon + \varepsilon. \end{aligned}$$

This mean that $\upsilon^*(Z_k) = 0$ and then $\upsilon(Z_k) = 0$ or $D_\upsilon G(x)$ exists allmost everywhere on E. ■

If $A \in S$ then we define shape of A as follow

$$r(A) = \frac{\sup\{\vartheta(N(x,l)): x \in A; N(x,l) \subset A)\}}{\inf\{\vartheta(N(x,l)): x \in A \subset N(x,l)\}}.$$

Further, if $E \in S, x \in E, \delta > 0, \rho > 0$ be given, a collection of all cell A with property $x \in A \subset N(x,\delta) \cap E$ and $r(A) > \rho$, denoted by $E(x,\delta,\rho)$.

Theorem 2.7: *Given a function $g: X \to Y$. If $g \in H(E,\vartheta)$ with ϑ −Henstock primitve of g on E is G, then for every real number $\varepsilon > 0$ there exist a real number $\gamma > 0$ such that for every interval $A \in I(E)$ with $\upsilon(A) < \gamma$ we have*

$$\|G(A)\| < \varepsilon.$$

Proof: Since G is a ϑ −Henstock primitve of g on E, then G is an additive function on $I(E)$. By Lemma 2.2 and Theorem 2.5, then for every real number $\varepsilon > 0$ there exist a real number $\gamma > 0$ such that for every interval $A \in I(E)$ with $\upsilon(A) < \gamma$ we have

$$\|G(A)\| < \varepsilon. \quad \blacksquare$$

3. CONCLUSION

If X is a locally compact metric space, Y is a Banach space, and G is a ϑ −Henstock primitive of a function $g: X \to Y$ on a cell E, then we have:

3.1 G is ϑ-continuous on $I(E)$,
3.2 G is strongly absolutely ϑ-continuous on $I(E)$ if g is a bounded on E,
3.3 $D_\upsilon G(x)$ exists allmost everywhere on E and $D_\upsilon G(x) = g(x)$ allmost everywhere on E,
3.4 for every real number $\varepsilon > 0$ there exist a real number $\gamma > 0$ such that for every interval $A \in I(E)$ with $\upsilon(A) < \gamma$ we have $\|G(A)\| < \varepsilon$.

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