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$\boldsymbol{C[a,b]}$-VALUED MEASURE AND SOME OF ITS PROPERTIES

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Abstract

Let $C[a,b]$ be the set of all real-valued continuous functions defined on a closed interval $[a,b]$. It is a commutative Riesz algebra space with unit element e, where $e(x)=1$ for every $x\in[a,b]$. As in the real numbers system $\mathbb{R}$, we define $\bar{C}[a,b]$ of the extended of $C[a,b]$. In this paper, we shall generalize the notions of outer measure, measure, measurable sets and measurable functions from $C[a,b]$ into $\bar{C}[a,b]$. This paper is a part of our study in Henstock-Kurzweil integral of functions define on a closed interval $[f,g]\subset C[a,b]$ which values in $\bar{C}[a,b]$.

Key words: outer measure, measure, measurable set, measurable function

INTRODUCTION

Some properties of real-valued continuous function defined on a closed interval were studied by several authors. Bartle and Sherbert [2] mention some of its properties are bounded, it has an absolute maximum and an absolut minimum, it can be approximated arbitrarily closely by step functions, uniformly continuous, and Riemann integrable.

In this paper, $C[a,b]$ denotes the set of all real-valued continuous functions defined on a closed interval $[a,b]$. Further discussion of $C[a,b]$ can be shown in classical Banach spaces such as Albiac and Kalton [1], Diestel [4], Lindenstrauss and Tzafriri [5], Meyer-Nieberg [6], and others.

In development of mathematical analysis, sometimes we need to extend of definition, such as measure. For example, Boccuto, Minotti and Sambucini [3] define Riemann sum of a function $\boldsymbol{f}:\boldsymbol{T}\to\boldsymbol{R}$, where $\boldsymbol{R}$ is a Riesz space, is

$$\boldsymbol{S(f,D)=\sum_{i=1}^{n} f(t_i)\mu(E_i)}$$

where $\boldsymbol{D=\{(E_i,t_i), i=1,2,\dots,n\}}$ is $\boldsymbol{\delta}$-fine partition of $\boldsymbol{T}$ and $\boldsymbol{\mu}$ is a Riesz-valued measure $\boldsymbol{R}$, that is $\boldsymbol{\mu:\Sigma\to R}$ where $\boldsymbol{\Sigma}$ is the $\boldsymbol{\sigma}$-algebra of all Borel subsets of $\boldsymbol{T}$. In their definition, they assumed that $\boldsymbol{R}$ is Dedekind complete Riesz space. Now, if we take $\boldsymbol{R=C[a,b]}$ that is Riesz space but not Dedekind complete, interesting for us to discuss a $\boldsymbol{C[a,b]}$-valued measure.

The aim of this paper is to construct a $\boldsymbol{C[a,b]}$-valued measure and to discuss some of its properties, including measurable sets and measurable functions. The construction of the $\boldsymbol{C[a,b]}$-valued measure could be applied to construct integral of $\boldsymbol{C[a,b]}$-valued functions.

PRELIMINARIES

Before we begin the discussion, we give an introductory about $C[a,b]$ as a Riesz space and a commutative Riesz algebra. Let $C[a,b]$ be the set of all real-valued continuous functions defined on a closed interval $[a,b]$. It is well known that $C[a,b]$ is a commutative algebra with unit element e, where $e(x)=1$ for every $x \in [a,b]$, over a field $\mathbb{R}$. If $f,g \in C[a,b]$, we define

$$f \le g \Leftrightarrow f(x) \le g(x)\,, \qquad f < g \Leftrightarrow f(x) < g(x) \quad \text{and} \quad f = g \Leftrightarrow f(x) = g(x)$$

for every $x \in [a,b]$. The relation " $\le$ " is a partial ordering in $C[a,b]$ because it satisfies

(i) $f \le f$ for every $f \in C[a,b]$,
(ii) $f \le g$ and $g \le h \Rightarrow f \le h$ for every $f,g,h \in C[a,b]$,
(iii) $f \le g$ and $g \le f \Rightarrow f = g$.

Therefore $(C[a,b], \le)$, briefly $C[a,b]$, is a partially ordered set. Further, the $C[a,b]$ satisfies

(i) $f \le g \Rightarrow f + h \le g + h$ for every $h \in C[a,b]$,
(ii) $f \le g \Rightarrow \alpha f \le \alpha g$ for every $\alpha \in \mathbb{R}^+$.

Therefore, $C[a,b]$ is also Riesz space. If $f,g \in C[a,b]$, we define fg with

$$(fg)(x) = f(x)g(x) \text{ for every } x \in [a,b].$$

Hence, $C[a,b]$ will be called a commutative Riesz algebra with unit element e. The Riesz spaces and commutative Riesz algebras more in-depth discussion can be found in [6] and [9].

So far, if $f,g \in C[a,b]$ with $f < g$, we define
$(f,g) = \{h \in C[a,b]: f < h < g\}$,
$[f,g] = \{h \in C[a,b]: f \le h \le g\}$,
$(f,\infty) = \{h \in C[a,b]: f < \infty\}$,
$[f,\infty) = \{h \in C[a,b]: f \le h\}$,
$(-\infty,g) = \{h \in C[a,b]: h < g\}$,
$(-\infty,g] = \{h \in C[a,b]: h \le g\}$.

If $f,g \in C[a,b]$, we define $f \vee g, f \wedge g, |f|, f^+, f^-$ with

(i) $(f \vee g)(x) = \sup_{x\in[a,b]}\{f(x), g(x)\}$,
(ii) $(f \wedge g)(x) = \inf_{x\in[a,b]}\{f(x), g(x)\}$,
(iii) $|f|(x) = |f(x)|$ for every $x \in [a,b]$,
(iv) $f^+(x) = \begin{cases} f(x), \text{if } f(x) \ge 0 \\ 0, \quad \text{if } f(x) < 0 \end{cases}$,
(v) $f^-(x) = \begin{cases} 0, \quad \text{if } f(x) \ge 0 \\ -f(x), \text{if } f(x) < 0 \end{cases}$.

Bartle and Sherbert [2] showed that if $f,g \in C[a,b]$, then $f \vee g$, $f \wedge g, |f|, f^+$ and f^- are members of $C[a,b]$. Explanation of infimum/supremum of set and limit of sequence on $C[a,b]$ can be shown in [8].

DISCUSSION

We shall construct a $\bar{C}[a,b]$-valued outer measure. We need an extention of the system of $C[a,b]$ as follows:

$$\bar{C}[a,b] = C[a,b] \cup \{-\infty, \infty\}$$

and we call it the **extended system of $\boldsymbol{C[a,b]}$**. If $f,g \in \bar{C}[a,b]$, we have $f+g \in \bar{C}[a,b]$ and $fg \in \bar{C}[a,b]$. The enlargement of the operations between $\pm\infty \in \bar{C}[a,b]$ and $f \in C[a,b]$ are defined as follows:

(i) $-\infty < f < \infty$ for every $f \in C[a,b]$,
(ii) $f+\infty = \infty$ and $f-\infty = -\infty$ for every $f \in C[a,b]$,
(iii) $f \cdot \infty = \infty$ and $f \cdot -\infty = -\infty$ for every $f \in C[a,b]$ and $f > \theta$,
(iv) $f \cdot \infty = -\infty$ and $f \cdot -\infty = \infty$ for every $f \in C[a,b]$ and $f < \theta$,
(v) $\infty + \infty = \infty$ and $-\infty + (-\infty) = -\infty$, and
(vi) $\theta \cdot \infty = \theta \cdot -\infty = \theta$

where θ is null element of $C[a,b]$ with $\theta(x) = 0$ for every $x \in [a,b]$.

Definition 1. *A function $\mu^*: 2^{C[a,b]} \to \bar{C}[a,b]$ is called a **$\boldsymbol{C[a,b]}$-valued outer measure**, briefly **outer measure**, if it satisfies the following properties:*

(i) $\mu^*(A) \geq \theta$ *for every* $A \in 2^{C[a,b]}$,
$\mu(\emptyset) = \theta$,
(ii) $A, B \in 2^{C[a,b]}$ *where* $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$, *and*
(iii) $\{A_n\} \subset 2^{C[a,b]} \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$.

If $f,g \in C[a,b]$ with $f < g$ and $I = (f,g)$ is a open interval, we defined a $C[a,b]$-valued interval function ℓ by

$$\ell(I) = g - f.$$

Next theorem is a example there is a outer measure on $C[a,b]$.

Theorem 2. *A function $\mu^*: 2^{C[a,b]} \to \bar{C}[a,b]$ that defined*

$$\mu^*(A) = \bigwedge\left\{\sum_{k=1}^{\infty} \ell(I_k): I_k \text{ is open interval for every } k \in \mathbb{N} \text{ and } A \subset \bigcup_{k=1}^{\infty} I_k\right\}$$

is a outer measure on $C[a,b]$.

Proof. It is clear that $\mu^*(A) \geq \theta$ for every $A \in 2^{C[a,b]}$. Since $\emptyset \subset A$ for every $A \in 2^{C[a,b]}$, then $\emptyset \subset (-\frac{e}{n}, \frac{e}{n})$ for every $n \in \mathbb{N}$. Therefore, we have

$$\mu^*(\emptyset) = \inf_{n} \ell\left((-\frac{e}{n}, \frac{e}{n})\right) = \inf_{n}\{\frac{2e}{n}\} = \theta.$$

Given two sets $A, B \subset C[a,b]$ arbitrary where $A \subset B$. If $\mu^*(B) = \infty$, it is true that $\mu^*(A) \leq \mu^*(B)$. If $\mu^*(B) < \infty$, then for every real number $\varepsilon > 0$ there is a sequence $\{(f_n, g_n)\} \subset C[a,b]$ such that $B \subset \bigcup_{n=1}^{\infty}(f_n, g_n)$ and $\sum_{n=1}^{\infty}(g_n - f_n) < \mu^*(B) + \varepsilon e$. Since $A \subset B$, then we have $A \subset \bigcup_{n=1}^{\infty}(f_n, g_n)$, hence $\mu^*(A) \leq \sum_{n=1}^{\infty}(g_n - f_n)$. Thus, we have

$$\mu^*(A) \leq \mu^*(B).$$

Given a sequence $\{A_n\} \subset C[a,b]$ arbitrary. If there is $m \in \mathbb{N}$ such that $\mu^*(A_m) = \infty$, it is true that $\mu^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$. If $\mu^*(A_n) < \infty$ for every $n \in \mathbb{N}$, then for every real number $\varepsilon > 0$ there is a sequence $\{(f_{n,k}, g_{n,k})\} \subset C[a,b]$ such that $A_n \subset \bigcup_{k=1}^{\infty}(f_{n,k}, g_{n,k})$ for every $n \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty}(g_{n,k}-f_{n,k})<\mu^*(A_n)+\frac{\varepsilon e}{2^n}.$$

For every $n\in\mathbb{N}$, we obtain $\bigcup_{n=1}^{\infty}A_n\subset\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{\infty}(f_{n,k},g_{n,k})$ and

$$\mu^*\left(\bigcup_{n=1}^{\infty}A_n\right)\le\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(g_{n,k}-f_{n,k})<\sum_{n=1}^{\infty}\left(\mu^*(A_n)+\frac{\varepsilon e}{2^n}\right)=\sum_{n=1}^{\infty}\mu^*(A_n)+\varepsilon e,$$

that is, $\mu^*(\bigcup_{n=1}^{\infty}A_n)\le\sum_{n=1}^{\infty}\mu^*(A_n)$. ■

Next, we know a $C[a,b]$-valued measure (the explanation of σ-algebra of set $\mathcal{A}$, measurable space $(X,\mathcal{A})$ and their properties can be shown in [7]).

Definition 3. *Let $X\subseteq C[a,b]$ be a nonempty set and $(X,\mathcal{A})$ be a measurable space. A function $\mu:\mathcal{A}\to\bar{C}[a,b]$ is called a **$C[a,b]$-valued measure on** $(X,\mathcal{A})$, briefly **measure on** X, if*

(i) $\mu(A)\ge\theta$ for every $A\in\mathcal{A}$
$\mu(\emptyset)=\theta$,
(ii) $\{A_n\}\subset\mathcal{A}$ where $A_m\cap A_n=\emptyset$ for $m\ne n$, then $\mu(\bigcup_{n=1}^{\infty}A_n)=\sum_{n=1}^{\infty}\mu(A_n)$

Let $X\subseteq C[a,b]$ be a nonempty set and μ a measure on measurable space $(X,\mathcal{A})$. A measure μ is called a **finite measure** if $\mu(X)<\infty$ and a measure μ is called a **σ-finite measure** if there is a sequence of measurable sets $\{X_n\}\subset\mathcal{A}$ such that $X=\bigcup_{n=1}^{\infty}X_n$ and $\mu(X_n)<\infty$ for every $n\in\mathbb{N}$. If μ is a measure on $(X,\mathcal{A})$, a pair $(X,\mathcal{A},\mu)$ is called a **measure space**. A measure space $(X,\mathcal{A},\mu)$ is called **complete** if $B\in\mathcal{A}$ with $\mu(B)=\theta$ and $A\subset B$ implies $A\in\mathcal{A}$. Some properties of measure on $(C[a,b],\mathcal{A})$ are given in Theorem 4, Theorem 5 and Theorem 6.

Theorem 4. *Let $(C[a,b],\mathcal{A},\mu)$ be a measure space. If $A,B\in\mathcal{A}$ and $A\subseteq B$ then $\mu(A)\le\mu(B)$.*

Proof. If $A\subseteq B$ then $B=A\cup(B-A)$ where $A\cap(B-A)=\emptyset$. So we have

$$\mu(B)=\mu(A)+\mu(B-A)\ge\mu(A).$$ ■

Theorem 5. *Let $(C[a,b],\mathcal{A},\mu)$ be a measure space. If $\{A_n\}\subset\mathcal{A}$ where $A_{i+1}\subseteq A_i$ for every $i\in\mathbb{N}$ and $\mu(A_1)<\infty$ then*

$$\mu\left(\bigcap_{n=1}^{\infty}A_n\right)=\lim_{n\to\infty}\mu(A_n)$$

if it has limit.

Proof. Set $A=\bigcap_{i=1}^{\infty}A_i$, we have $A_1=A\cup(\bigcup_{i=1}^{\infty}(A_i-A_{i+1}))$ with $A\cap(A_i-A_{i+1})=\emptyset$ for every $i\in\mathbb{N}$. Then $\mu(A_1)=\mu(A)+\sum_{i=1}^{\infty}\mu(A_i-A_{i+1})$ and $A_{i+1}\cap(A_i-A_{i+1})=\emptyset$, we have

$$\mu(A_i-A_{i+1})=\mu(A_i)-\mu(A_{i+1}).$$

If $\lim_{n\to\infty}\mu(A_n)$ exist, we have

$$\begin{aligned}\mu(A_1)&=\mu(A)+\sum_{i=1}^{\infty}(\mu(A_i)-\mu(A_{i+1}))=\mu(A)+\lim_{n\to\infty}\sum_{i=1}^{n-1}(\mu(A_i)-\mu(A_{i+1}))\\&=\mu(A)+\mu(A_1)-\lim_{n\to\infty}\mu(A_n),\end{aligned}$$

that is, $\mu(\bigcap_{n=1}^{\infty}A_n)=\mu(A)=\lim_{n\to\infty}\mu(A_n)$. ■

Theorem 6. *Let $(C[a,b],\mathcal{A},\mu)$ be a measure space. If $\{A_n\}\subset\mathcal{A}$ then*

$$\mu\left(\bigcup_{n=1}^{\infty}A_n\right)\leq\sum_{n=1}^{\infty}\mu(A_n).$$

Proof. Set $B_1=A_1$ and $B_n=A_n-(\bigcup_{i=1}^{n-1}A_i)$ for every $n\geq 2$. Then $B_n\subseteq A_n$ for every n and $B_i\cap B_j=\emptyset$ for every $i\neq j$. Thus $\mu(B_n)\leq\mu(A_n)$ for every n and

$$\mu\left(\bigcup_{n=1}^{\infty}A_n\right)=\sum_{n=1}^{\infty}\mu(B_n)\leq\sum_{n=1}^{\infty}\mu(A_n).$$ ■

Measurable Sets

We introduce a definition of a μ^*-measurable set.

Definition 7. *A set $E\subset C[a,b]$ is said **μ^*-measurable** if every $A\subseteq C[a,b]$ we have*

$$\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap E^C).$$

Some μ^*-measurable sets on $C[a,b]$ is given in Theorem 8 as follows.

Theorem 8. *The following statements are true:*

(i) *$\emptyset$ and $C[a,b]$ are μ^*-measurable,*
(ii) *If $E\subset C[a,b]$ is μ^*-measurable, then E^C is μ^*-measurable,*
(iii) *If $E_1,E_2\subset C[a,b]$ is μ^*-measurable, then $E_1\cup E_2$ and $E_1\cap E_2$ are μ^*-measurable.*

Proof. We only prove (iiii). Consider $A\cap(E_1\cup E_2)=(A\cap E_1)\cup\left(A\cap E_2\cap E_1^C\right)$ that implies

$$\mu^*\left(A\cap(E_1\cup E_2)\right)\leq\mu^*(A\cap E_1)+\mu^*\left(A\cap E_2\cap E_1^C\right) \qquad \text{(a)}$$

Since E_2 is μ^*-measurable set, then we have

$$\mu^*\left(A\cap E_1^C\right)=\mu^*\left(\left(A\cap E_1^C\right)\cap E_2\right)+\mu^*\left(\left(A\cap E_1^C\right)\cap E_2^C\right) \text{ or}$$

$$\mu^*\left(\left(A\cap E_1^C\right)\cap E_2\right)=\mu^*\left(A\cap E_1^C\right)-\mu^*(A\cap(E_1\cup E_2)^C). \qquad \text{(b)}$$

Subtitution (b) into (a), we have

$$\mu^*\left(A\cap(E_1\cup E_2)\right)+\mu^*(A\cap(E_1\cup E_2)^C)\leq\mu^*(A\cap E_1)+\mu^*\left(A\cap E_1^C\right)=\mu^*(A).$$

Thus, $E_1\cup E_2$ is μ^*-measurable set. Based on this result and a statement (ii), we have $E_1\cap E_2=\left(E_1^C\cup E_2^C\right)^C$ μ^*-measurable. ■

Theorem 9. *If $E_1,E_2,\ldots,E_n\subset C[a,b]$ are disjoint and μ^*-measurable sets, then for every $A\subset C[a,b]$ we have*

$$\mu^*\left(A\cap\bigcup_{k=1}^{n}E_k\right)=\sum_{k=1}^{n}\mu^*(A\cap E_k).$$

Proof. We shall prove by mathematical induction.
That is clear true for $n=1$. Next, we assume the theorem is true for $E_1,E_2,\ldots,E_{n-1}$ sets, that is

$$\mu^*\left(A \cap \bigcup_{k=1}^{n-1} E_k\right) = \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$$

is true for every $A \subset C[a,b]$.

Since E_k disjoint sets, we have

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n = A \cap E_n$$

and

$$A \cap \left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^C = A \cap \bigcup_{k=1}^{n-1} E_k \,.$$

Since E_n is μ^*-measurable set for every n, we have

$$\mu^*\left(A \cap \bigcup_{k=1}^{n} E_k\right) = \mu^*(A \cap E_n) + \mu^*\left(A \cap \bigcup_{k=1}^{n-1} E_k\right)$$

$$= \mu^*(A \cap E_n) + \sum_{k=1}^{n-1} \mu^*(A \cap E_k)$$

$$= \sum_{k=1}^{n} \mu^*(A \cap E_k)\,.$$ ■

Theorem 10. *If* $\{E_n\}$ *is a* μ^**-measurable sets sequence, then*

$$\bigcup_{n=1}^{\infty} E_n$$

is a μ^**-measurable set.*

Theorem 11. *If* $\mathcal{A}$ *is a collection of all* μ^**-measurable sets on* $C[a,b]$, *then* $\mathcal{A}$ *is a* σ*-algebra on* $C[a,b]$.

Proof. Since $\emptyset \in \mathcal{A}$ then $\mathcal{A} \neq \emptyset$. Based on definition, if $E \in \mathcal{A}$ we have $E^C \in \mathcal{A}$, and based on Theorem 10, if $E_n \in \mathcal{A}$ then $\bigcup_{n=1}^{\infty} E_n$ is μ^*-measurable set. ■

Let $X \subset C[a,b]$ be a nonempty set. If $\mathcal{A}$ is a collection of μ^*-measurable subsets of $C[a,b]$, we have a measurable space $(X,\mathcal{A})$ that is generated by a measure μ^* as defined in Theorem 2.

Theorem 12. *Let* $X \subset C[a,b]$ *be a nonempty set and* $(X,\mathcal{A})$ *be a measurable space. A function* $\mu: \mathcal{A} \to \bar{C}[a,b]$, *formulated by*

$$\mu(E) = \mu^*(E),$$

is a measure.

Proof. $\mu(E) = \mu^*(E) \geq \theta$ for every $E \in \mathcal{A}$ and $\mu(\emptyset) = \mu^*(\emptyset) = \theta$. If $\{E_k\} \subset \mathcal{A}$ and E_k are disjoint sets, with Theorem 9 and replace $A = C[a,b]$ we obtain

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k) = \sum_{k=1}^{n} \mu(E_k)$$

for every $n \in \mathbb{N}$. ∎

Measurable Functions

Before we discuss a measurable function on a μ^*-measurable set, we introduce a characteristic function and a simple function on $C[a,b]$.
Let $(C[a,b], \mathcal{A}, \mu)$ be a measure space and $E \subset C[a,b]$. A function

$$\chi_E: C[a,b] \to \bar{C}[a,b]$$

is called **characteristic function** on E if

$$\chi_E(h) = \begin{cases} e & , h \in E \\ \theta & , h \notin E \end{cases}$$

If $E_1, E_2, \dots, E_n$ are μ^*-measurable sets and $c_1, c_2, \dots, c_n \in \mathbb{R}$, a function

$$\varphi = \sum_{k=1}^{n} c_k \chi_{E_k}$$

is called a **simple function** on $E = \bigcup_{k=1}^{n} E_k$ μ^*-measurable set. A simple function φ is said in the form **canonical representation** if $E_1, E_2, \dots, E_n$ are disjoint sets. Every simple function always can be represented in canonical representation. In what follows, we shall always assume that every simple function is in the form of canonical representation if there is nothing further information. If φ and ψ are simple functions on set E, then $\alpha\varphi$ and $\varphi + \psi$ are simple functions on set E for every $\alpha \in \mathbb{R}$.

Before defining a measurable function on a μ^*-measurable set, we need a terminology what is called “almost everywhere”. Let $P(h)$ denote a statement concerning the points h in a set E. We say that the statement $P(h)$ holds true **almost everywhere** on E or $P(h)$ holds true **for almost** every $h \in E$ if there is $A \subset E$ with $\mu^*(A) = \theta$ such that $P(h)$ holds true for every $h \in E - A$.

Definition 13. *A function $F: E \subseteq C[a,b] \to \bar{C}[a,b]$ is said to be* ***measurable*** *on a μ^*-measurable set E if for every number $\varepsilon > 0$ there is a simple function φ on E such that*

$$|F(h) - \varphi(h)| < \varepsilon e$$

almost everywhere on E.

By the definition, it is clear that every simple function on μ^*-measurable set E is measurable on E.

Theorem 14. *If $F, G: E \subseteq C[a,b] \to \bar{C}[a,b]$ are two functions such that F and G are measurable on μ^*-measurable set E, then for every number $\alpha \in \mathbb{R}$ functions αF and $F + G$ are measurable on μ^*-measurable set E.*

Corollary 15. *If $F_i: E \subseteq C[a,b] \to \bar{C}[a,b]$ ($i = 1,2, \dots, n$) are functions such that F_i is* *measurable on μ^*-measurable set E for every $i = 1,2,\dots,n$, then for every $\alpha_i \in \mathbb{R}$ $(i = 1,2,\dots,n)$, a function*

$$\sum_{i=1}^{n} \alpha_i F_i$$

is measurable on μ^*-measurable set E.

Teorema 16. *If $F: [f,g] \to \bar{C}[a,b]$ is continuous function, then F is measurable on $[f,g]$.*

Proof. Interval $[f,g]$ is μ^*-measurable set. If F is continuous function on closed interval $[f,g]$ then F is uniformly continuous on $[f,g]$, that is, for every number $\varepsilon > 0$ there is number $\eta > 0$ such that for every $s,t \in [f,g]$ with $|s-t| < \eta e$ we have $|F(s) - F(t)| < \varepsilon e$. Let $\{[f_0, f_1], [f_1, f_2], \dots, [f_{n-1}, f_n]: f_0 = f_1 \text{ and } f_n = g\}$ be a partition on $[f,g]$ such that $f_k - f_{k-1} < \varepsilon e$ for every $k = 1,2,\dots,n$. If we take $A_k = [f_{k-1}, f_k]$ and $a_k = \sup_{x\in[a,b]}\{F(f_k)(x)\} \in \mathbb{R}$ for every k, then we obtain a simple function

$$\varphi = \sum_{k=1}^{n} c_k \chi_{A_k}$$

on $[f,g]$ such that

$$|F(h) - \varphi(h)| < \varepsilon e$$

for every $h \in [f,g]$, that is, F is measurable on $[f,g]$. ■

CONCLUSION AND SUGGESTION

From the discussion results above, we conclude:

1. There is a $C[a,b]$-valued measure that is generated by $C[a,b]$-valued outer measure.
2. A continuous function that defined on a closed interval subset of $C[a,b]$ is measurable on it.

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