PE - 07

TREATING FOURIER SERIES AS VECTOR: A CONCEPT SIMPLIFICATION FOR TEACHING FOURIER SERIES

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Abstract: Fourier series as a representation of periodic function is quite frequently encountered in various fields of science and engineering, especially in physics. In practice, even non-periodic function defined in a certain range of the domain can be represented by Fourier series. Vectors, on the other hand, are usually introduced earlier than Fourier series. Similar to Fourier series, vectors are also used intensively and extensively in physics. In this paper we will examine the properties of Fourier series that account for its similarities to the properties of vectors. The orthogonal properties of bases vectors in the context of dot product amongst bases vectors will be the central discussion to examine the similarities between vectors and Fourier series.
Keywords: Fourier series, vectors, dot product, orthogonal

INTRODUCTION

The progress of science and technology seem to increase exponentially with respect to time. On the other hand, the capability of students in comprehending the science learning seems to be stagnant. Concerning the limitation of allocated time for learning, there must be some ways to manage the learning concepts. One of the methods that can be used is analogy. Through analogy new learning materials that have similar characteristics with some of the existing knowledge can be integrated into a larger piece of knowledge with much less effort. Another advantage of this approach is that the new acquired information will last longer in the students' memory.

Fourier series is one of the main topics in advanced mathematical physics taught in every Physics Department. From the author's observation, most physics students found difficulties in understanding the essential concept of Fourier series. Finding the coefficients of a Fourier series representing a periodic function seems to be a heavy job to be done. This happens because most of the physics students do not try to explore their existing knowledge that has similar characteristics with the new knowledge (Fourier series). They tend to memorize the formula without knowing how and why. The extensive use of Fourier series in various fields of science drives the author's motivation to explore the similarity characteristics of Fourier series with other mathematical topics commonly known by students.

Most of physics students are quite familiar with vectors. They have been acquainted with vectors since they were in high school. The extensive use of vectors in various applications, especially in physics, strengthens their comprehension. The author will employ this fact as the starting point to study the similarity between vectors and Fourier series. In this study the author will limit the scope of discussion in the physics application of Fourier series and vectors.

REVIEWS ON VECTORS

Any physical quantity that has magnitude and direction is generally known as vector quantity. In most applications, vectors generally have units or dimensions attached to its magnitude. Conventionally we use vector symbol $\vec{A}$ or simply $\boldsymbol{A}$ (Spiegel, Lipschutz, and Spellman, 2009) to identify a physical quantity as a vector quantity. The complete description can be written as

$$\vec{A} = A\ \hat{e}_A \tag{1}$$

The first factor in equation (1) is the magnitude of $\vec{A}$, and the second factor is the unit vector in the direction of $\vec{A}$. As suggested by its name, the unit vector is a vector with a unit magnitude, but without physical unit (dimension). Unit vector functions as directional information of the associated vector. In many cases, it is more convenient to write a vector as the sum of the vector components in the axial directions of the space, for example in the X, Y, and Z axis. But, in more general condition, the axial directions of the space can be any number N, and we can still assume the orthogonality of the system. Assuming the latter, equation (1) can be written as

$$\vec{A} = \sum_{i=1}^{N} A_i\ \hat{e}_i \tag{2}$$

where $\hat{e}_1, \hat{e}_2, \cdots, \hat{e}_N$ are, respectively, the unit vectors in the axial directions of the N-dimensional space. Those unit vectors are called the basis vectors that span the N-dimensional vector space. They form a complete set and are orthogonal to each other. Therefore, they satisfy the following relationships

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}\ ; \qquad i,j = 1, 2, \cdots, N \tag{3}$$

where δ_{ij} is the Kronecker delta (Lai *et.al*, 2009) that has the value of 1 if $i = j$ and 0 if $i \neq j$. $A_1, A_2, \cdots, A_N$ in equation (2) are, respectively, components of $\vec{A}$ in the first, second, … , and the N-th axial direction of the space. Comparing equation (1) and equation (2) and using the relationships in equation (3) we may conclude that the squared magnitude of $\vec{A}$ can be written as

$$A^2 = \sum_{i=1}^{N} A_i^2 \tag{4}$$

Likewise, the n-th component of $\vec{A}$ can be obtained by performing dot product with $\hat{e}_n$ on both sides of equation (2). Using the orthogonal property of the basis vectors as shown in equation (3) we obtain

$$\vec{A} \cdot \hat{e}_n = A_n \tag{5}$$

TREATING FOURIER SERIES AS VECTOR

Any periodic well-behaved function with basic period of 2π can be represented by a series of the form

$$f(x) = \frac{1}{2}\, a_0 + \sum_{n=1}^{\infty} \{ a_n \cos(nx) + b_n \sin(nx) \} \tag{6}$$

which is known as Fourier series (Brown and Churchill, 2011). As a representation of a function, Fourier series guarantee that the value of the series at any values of x in the range of basic period always converges to the value of the associated function at the corresponding x. The problem frequently encountered by students is to find the coefficients of the Fourier series and how to interpret the result. Apparently most of the students were drowned in the technical details of the calculation that frequently involves tedious integration processes.

If we consider the set of $\cos(nx)$ and $\sin(nx)$ for $n = 0, 1, 2, \cdots$ as the basis functions on the basic period of 2π, then we see the resemblance between Fourier series on the right side of equation (6) and the sum of vector components on the right side of equation (2). The similarity between the two will even be more pronounced if we denote the basis functions $\cos(nx)$ and $\sin(nx)$ for $n = 0, 1, 2, \cdots$ by, respectively, $\alpha_n(x)$ and $\beta_n(x)$. To get further similarity we have to define "dot product" between two functions in the functional space. This will be done by multiplying the two functions then calculate its integral within an interval of 2π. For example, the "dot product" between any pair of basis functions can be written as

$$\begin{aligned} \int_{-\pi}^{\pi} \alpha_m(x) \cdot \alpha_n(x)\ dx &= \begin{cases} 2\pi\, \delta_{mn} & if\ m = 0 \\ \pi\, \delta_{mn} & if\ m \neq 0 \end{cases} \\ \int_{-\pi}^{\pi} \beta_m(x) \cdot \beta_n(x)\ dx &= \pi\, \delta_{mn} \\ \int_{-\pi}^{\pi} \alpha_m(x) \cdot \beta_n(x)\ dx &= 0. \end{aligned} \tag{7}$$

It turns out that the chosen basis functions form a complete orthogonal set. The only difference between the relationships in equation (3) and in equation (7) is the magnitude of the dot product between a basis function and itself. This difference should be accepted as a consequence of the choice of the basis functions. Similar does not mean identical in all aspects; therefore, we have to accept this difference as is.

The next step to complete the Fourier series is to calculate its coefficients. Comparing equation (6) with equation (2) we found out that the Fourier coefficients a_n and b_n are analogous to the component of a vector. Therefore, following the same argument for finding a vector component we will obtain

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \alpha_n(x)\ dx; \quad n = 0, 1, 2, \cdots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cdot \beta_n(x)\ dx; \quad n = 1, 2, 3, \cdots \end{aligned} \tag{8}$$

The index of the "a" coefficients start from zero, while the index of the "b" coefficients start from one because the $\alpha_n(x)$ or $\cos(nx)$ basis functions start from $n = 0$, and the $\beta_n(x)$ or $\sin(nx)$ basis functions start from $n = 1$. The factor ½ associated with a_0 in equation (6) is used for convenience. The advantage of using this factor is that we can get the same formula to calculate a_n for any values of n. Otherwise, we have to use different formula to calculate a_n for $n = 0$, and for $n \neq 0$; which is very inconvenient.

DISCUSSION

Mastering new materials will need more time and efforts unless we have sufficient familiarity with the characteristics of the new materials. Prior knowledge in vectors is very useful for understanding the basic concepts of Fourier series. Exploring the analogy between vectors and Fourier series will help students expand their body of knowledge by integrating and internalizing the new materials (Fourier series) with their existing knowledge (vectors). The students gain new understanding on vectors in interpreting Fourier series as a "*higher order*" vector. Understanding the new expanded concept will free students from memorizing too many formulae. Instead of recalling, students can use their comprehension to find the components of the Fourier series based on their knowledge of orthogonality and generalized dot product.

By analogy, we have treated Fourier series as vector of infinite dimension or, more precisely, as the sum of infinite vector components by employing cos $(\boldsymbol{nx})$ and sin $(\boldsymbol{nx})$ as basis vectors. An advantage of the basis functions choice is that they are orthogonal as shown in equation (7). Therefore, determining the Fourier coefficient can be obtained by analogy as in vectors: performing "dot product" with the corresponding basis function on both sides of the equation. Moreover, we may proceed to explore the analogy of the squared magnitude of the vector. Performing "dot product" with itself on both sides of equation (6), after applying the orthogonal property of the basis functions in equation (7), we obtain

$$\int_{-\pi}^{\pi} f(x) \cdot f(x)\, dx \;=\; \pi \left\{ 2 \cdot \left(\frac{1}{2}\, a_0 \right)^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right\} \tag{9}$$

In the discussion of Fourier series, equation (9) is usually written in a slightly different form:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \big(f(x)\big)^2\, dx \;=\; \left(\frac{1}{2}\, a_0 \right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)\,, \tag{10}$$

and is known as the Parseval's theorem (Boas, 2005).

Of course there still remain abundant details to be done to get the explicit solution of Fourier series representing a given function. It is unavoidable, but it is not the topic in this discussion. The main purpose of this study is not dealing with the detail calculation, but how to gain new understanding by integrating the new knowledge with the existing knowledge having similar characteristics with the new one. By doing this students will spend less time and efforts to understand the new knowledge.

CONCLUSION

Fourier series and vectors have many similar characteristics. By analogy Fourier series can be regarded as vectors of infinite dimension. The most advantage of treating Fourier series as vectors is new understanding on the generalized vectors. No need to memorize the formulae for determining neither the Fourier coefficients nor the Parseval's theorem. It will come naturally from the understanding of orthogonality of the basis functions.

ACKNOWLEDGMENT
The author would like to express his gratitude to Prof. Sugimin W.W. for the intensive discussion that initiated this study. He would also like to extend his gratefulness to the Physics Education Department of Widya Mandala Catholic University Surabaya for the unfailing support to this study.

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